On the Classification and Invariantive Characterization of Nilpotent Algebras.[*]

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CHAPTER I.

INTRODUCTION.

1. *Relation of Problem to Classification and Invariantive Characterization of General Linear Algebras.* Linear associative algebras in a small number of units, with coordinates ranging over the field C of ordinary complex numbers, have been completely tabulated; that is, their multiplication tables have been reduced to very simple forms.† But if we had before us a linear associative algebra, the chances are that its multiplication table would not be in any of the tabulated forms, nor even in such a form that we could readily ascertain to which standard form it was equivalent. Accordingly the question naturally arises: "Can we find invariantive criteria which will tell us when two algebras are equivalent? or, as we say, which will completely characterize the algebras?"

In a previous paper ‡ we considered the problem of finding invariants which would completely characterize linear associative algebras in two or three units with a modulus, over the field C. The terms "invariants" and "characterize," be it understood, are used here in the sense defined by Professor Dickson. § For these special cases, the algebras || can be completely characterized by invariants obtained by the application of the following theorem: "In a general n-ary linear algebra over any field F, both characteristic determinants are absolute covariants of the algebra; their coefficients are absolute covariants; and the invariants and covariants of the characteristic determinants

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† See especially B. Peirce, AMERICAN JOURNAL OF MATHEMATICS, Vol. IV (1881), pp. 97–192; Study, *Göttinger Nachrichten*, 1889, pp. 237–268; G. Scheffers, *Mathematische Annalen*, Vol. XXXIX (1891), pp. 293–390; Hawkes, *Mathematische Annalen*, Vol. LVIII (1904), pp. 361–379; AMERICAN JOURNAL OF MATHEMATICS, Vol. XXVI (1904), pp. 223–242.

‡ Hazlett, *Annals of Mathematics*, Vol. XVI, p. 1.

§ AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXI, p. 337. See § 6 of this paper.

|| For brevity, we will usually refer to linear algebras simply as algebras; but, unless specified, we do not assume the associative law nor the commutative law.

and of the coefficients of the powers of ω in these determinants are respectively invariants and covariants of the linear algebra."

If now we proceed to the characterization of the quaternary associative algebras with a modulus, we find that the invariants given us by this theorem only partially characterize these algebras. In other words, the general problem can not be solved by use of this theorem alone.

But fortunately Cartan* and Wedderburn† have some general theorems which bear on the problem.‡ These tell us that we can characterize the general algebras if we can characterize three special kinds of algebras; namely, simple matric algebras, division algebras, and nilpotent algebras. Now in a given number of units there is only one simple matric algebra. Over the field C, there is no division algebra other than the algebra of ordinary complex numbers; and in general, the Galois Fields are the only algebras of a finite number of elements such that every number except zero has an inverse, § whereas, over the field C, there is an infinite number of classes of nilpotent algebras.

In this paper, we shall see how nilpotent algebras can be characterized by the aid of certain homogeneous polynomials whose coefficients are constants of multiplication. For algebras in a small number of units, these polynomials are sufficient if we assume the commutative and associative laws; but to characterize the general nilpotent algebra, we need further invariants.

2. *Definitions for Linear Algebras.* If we have given a set of *units* $e_1, \ldots, e_n$ linearly independent with respect to a field F, and such that

$$e_i e_j = \sum_{k=1}^{n} \gamma_{ijk} e_k \quad (i, j = 1, \ldots, n), \tag{1}$$

where the *constants of multiplication* γ_{ijk} range over F; and if the complex numbers of the form $X = \sum_{i=1}^{n} x_i e_i$ combine under addition and subtraction as follows:

$$X \pm Y = \sum_{i}^{1, n} (x_i \pm y_i) e_i,$$

and if they combine under multiplication according to the distributive law, then the set of complex numbers is said to form a *linear algebra* ‖ over the field F with multiplication table (1). The *characteristic right-hand and left-hand determinants* are respectively defined to be

* *Annales de Toulouse*, Vol. XII (1898) B.

† *Proceedings of the London Mathematical Society*, Series 2, Vol. VI (1908), pp. 77–118.

‡ See end of § 2.

§ Wedderburn, *Transactions of the American Mathematical Society*, Vol. VI (1905), p. 349.

‖ For other definitions, see Dickson, *Transactions of the American Mathematical Society*, Vol. IV (1903), pp. 21–26.

$$\left.\begin{aligned}\delta(\omega) &\equiv |\sum_i \gamma_{ijk} x_i - d_{jk}\omega|,\\ \delta'(\omega) &\equiv |\sum_i \gamma_{jik} x_i - d_{jk}\omega|.\end{aligned}\right\} \tag{2}$$

There may exist in the algebra a number ε called a *principal unit* or *modulus*, such that, for every number x in the algebra,

$$x\varepsilon = x = \varepsilon x.$$

If such a number exists, it is unique.

Division, as a rule, is not unique; that is, if we know the product of two numbers is zero, it does not necessarily follow that one of the numbers is zero. An algebra in which right- and left-hand division, except by zero, is always possible and unique is called a *division algebra* by Dickson,* a *primitive algebra* by Wedderburn.†

Another important family of algebras are the *simple matric algebras*;‡ that is, those which have the multiplication table of the form

$$e_{pq}e_{st} = d_{qs}e_{pt} \quad (p, q, s, t = 1, \ldots, n),$$

where the units are e_{ij} $(i, j = 1, \ldots, n)$.

The *complex*§ $A = (x_1, x_2, \ldots, x_a)$ is defined as the set of all quantities linearly dependent on $x_1, x_2, \ldots, x_a$, and the number of linearly independent elements is called the *order* of the complex. If A and B are two complexes, the complex formed by all quantities linearly dependent on the elements of A and B is called the *sum* of A and B, and is denoted by $A+B$. If a complex C is contained in the complex A, we write $C \leq A$ or $A \geq C$; similarly, if x is an element of the complex A, we write $x < A$. The elements common to two complexes also form a complex. The greatest complex common to two complexes A and B is denoted by $A \cap B$.

If A and B are any two complexes, and if x and y are any elements of A and B respectively, the complex of all elements linearly dependent on those of the form xy is called the *product* of A and B, and is written AB. If multiplication of all the elements involved is associative, then multiplication of complexes is likewise associative. If the associative law of multiplication is assumed, then the integral powers of a complex A are defined by means of the recursion formula $A \cdot A^n = A^{n+1} = A^n \cdot A$. In general, however, the integral powers of a complex A are defined by the formula

$$A^{m+1} = \sum_{i=1}^{m} A^i A^{m+1-i}.$$

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* "Linear Algebras," *Cambridge Tracts in Mathematics and Mathematical Physics*, 1914, p. 66.

† *Loc. cit.*, p. 91.

‡ Wedderburn, *loc. cit.*

§ The definitions in the next two paragraphs are those of Wedderburn, *loc. cit.*, pp. 79–80. For this notion of "complex" compare Kronecker, *Berliner Sitzungsberichte*, 1888, p. 597.

The necessary and sufficient condition that the complex A be an algebra is that $A^2 < A$.

To illustrate these definitions, consider the algebra of four units

$$(e_0, e_1, e_2, e_3),$$

quaternions, over the field F, whose multiplication table is

$$e_1 e_2 = -e_2 e_1 = e_3, \quad e_2 e_3 = -e_3 e_2 = e_1, \quad e_3 e_1 = -e_3 e_1 = e_2,$$
$$e_i^2 = -e_0 \ (i=1,2,3), \quad e_j e_0 = e_j = e_0 e_j \ (j=0,1,2,3).$$

If $x_0, x_1, \ldots, y_0, y_1, \ldots$ be marks of F, then the totality of all numbers of the form $x_1 e_1 + x_2 e_2 + x_3 e_3$ form a complex A of order 3, but they do not form an algebra. The totality of all numbers of the form $y_0 e_0 + y_1 e_1$ form a complex B of order 2, and they also form an algebra.

$$A+B = (e_0, e_1, e_2, e_3) = AB = A^2 = A^3; \ B^2 = B; \ A \cap B = (e_1).$$

A subcomplex B of a complex A which is such that $AB \leq B$ and $BA \leq B$ is called an *invariant* * subcomplex of A. B is necessarily an algebra. An algebra which has no invariant subcomplex is said to be *simple.* †

If an algebra A is expressible as the sum of two algebras $A_1 \neq 0$ and $A_2 \neq 0$ which are such that $A_1 A_2 = 0 = A_2 A_1$, then A is said to be *reducible* and to be the *direct sum* ‡ of A_1 and A_2. A_1 and A_2 can be so chosen that $A_1 \cap A_2 = 0$. Similarly for a complex A.

If C and D are any two algebras, such that every element of one is commutative with every element of the other, and if further the order of $A = CD$ is the product of the orders of C and D, then the algebra A is said to be the *direct product* § of C and D.

If an algebra A have a modulus, then $A^2 = A$; and in general, since we are dealing with algebras having a finite basis, we must have $A^{a+1} = A^a$ for some integer.

In particular, A may be such that $A^a = 0$; if so, A is said to be *nilpotent.* The smallest such integer is called the *index* of the algebra.‖ For a nilpotent algebra, $\delta(\omega) \equiv \omega^n \equiv \delta'(\omega)$. For associative algebras this definition of nilpotent algebras is equivalent to that given by Cartan and others;¶ but for non-associative algebras the two definitions are not equivalent, as can be seen by the

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* Cartan, *loc. cit.*, p. 57; Molien, *Mathematische Annalen*, Vol. XLI (1893); Frobenius, *Berliner Sitzungsberichte*, 1903, p. 523.

† Cartan, *loc. cit.*, p. 57.

‡ "Sum" was first used in this sense by Scheffers, *loc. cit.*, p. 323.

§ Wedderburn, *loc. cit.*, p. 99.

‖ Wedderburn, *loc. cit.*, p. 87.

¶ Wedderburn, *loc. cit.*, pp. 88–91.

following example, which is nilpotent in Cartan's sense, but not in Wedderburn's, $e_1 e_2 = -e_2 e_1 = e_3$, $e_2 e_3 = -e_3 e_2 = e_1$, $e_3 e_1 = -e_1 e_3 = e_2$.

An algebra which has no nilpotent invariant subalgebra is called *semi-simple*.* Such an algebra always has a modulus. Cartan† has shown that an associative algebra A, with a modulus, over the field C of ordinary complex numbers is the sum of a semi-simple algebra S and a nilpotent algebra N, such that every unit of N is commutative with every unit of S.

Wedderburn‡ has proved a similar theorem for associative algebras for a general field F; namely: "Any associative algebra can be expressed as the sum of a nilpotent algebra and a semi-simple algebra. The latter algebra is not unique, but any two determinations of it are simply isomorphic. Furthermore, a semi-simple associative algebra can be expressed uniquely as the direct sum of a number of simple algebras; and a simple associative algebra can be expressed as the direct product of a primitive algebra and a simple matric algebra."

CHAPTER II.

General Theory.

Preliminary Theorems, §§ 3–8.

3. *Canonical Form for Nilpotent Algebras.* Wedderburn has the following theorem which gives a very simple form to which any nilpotent algebra can be transformed.

Theorem. *If E be a nilpotent algebra (not necessarily associative) with index α, then if $B \equiv E$ (mod E^2),§ $E = \sum_{i}^{1,\alpha-1} B^i$.*

For since $E = B + E^2$, we have $E^2 \leq B^2 + E^3$ and thus $E = B + B^2 + E^3$. By induction, we have in general $E^h \leq B^h + E^{h+1}$ and thus $E = \sum_{i=1}^{\alpha-1} B^i + E^\alpha$. Moreover, for any positive integer p, $E^p = \sum_{k=p}^{\alpha-1} B^k$.

If the multiplication table of a nilpotent algebra be in the form given by this theorem, the algebra will be said to be in its *canonical form*. Furthermore, if an algebra be expressible in this form, then it is nilpotent in the sense defined by Wedderburn; and hence, if multiplication be associative, it is also nilpotent in the sense defined by Cartan and others.

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* This is the definition given by Wedderburn, *loc. cit.*

† *Loc. cit.*, p. 50.

‡ Wedderburn, *loc. cit.*, pp. 86, 94.

§ That is, B is derived from E by considering as equal those elements of E which differ only by an element of E^2.

4. *The Group Which Leaves the Canonical Form Unaltered.* Consider a nilpotent algebra E over the field F in canonical form, $E = \sum_{i=1}^{\alpha-1} B^i$, where $B \equiv E \pmod{E^2}$ and where α is the index of E. If we have a second algebra E_1 over the field F and in canonical form, $E_1 = \sum_{i=1}^{\alpha-1} B_1^i$, where $B_1 \equiv E_1 \pmod{E_1^2}$, then if E is equivalent to E_1, $B_1 \leq B + E^2$. Therefore $B_1^2 \leq B^2 + E^3$; and in general, by induction, we have $B_1^m \leq B^m + E^{m+1}$. Hence, if we denote by $e_i^{(p)}$ a unit of E which is in B^p, and similarly for E_1, then the transformation which carries E into E_1 is of the form

$$e_i'^{(p)} = \sum_{q=p}^{\alpha-1} \sum_j a_{ij}^{(p,q)} e_j^{(q)}, \tag{3}$$

where the a's are in F. We shall use G to denote the group of all such transformations.

5. *Invariance of Order of B^a (mod E^{a+1}). Modification of Canonical Form.* The equation $E = \sum_{i=1}^{\alpha-1} B^i$ means only that every number of E can be expressed as a linear homogeneous function of elements in $B, \ldots, B^{\alpha-1}$; two of these complexes may overlap, as in the quaternary algebra $e_1e_1 = e_2$, $e_1e_2 = e_3$, $e_2e_1 = e_1e_3 = e_4$, where the products not written are zero. Moreover, for $1 < i < \alpha$, the number of linearly independent elements in B^i depends on the particular choice of B. But the number of linearly independent elements in B is an invariant; and in general, for each a, the number of linearly independent elements in B^a reduced modulo E^{a+1} is an invariant.

To show this, consider a fixed a, and subject the units of the algebra to a transformation T of the group G. Now $T = T_1T_2$, where T_1 transforms the units of each power of B among themselves alone, and where T_2 at most adds to each unit of B^i some element of E^{i+1}, for every i. Clearly T_1 does not change the number of linearly independent elements in $B^a \pmod{E^{a+1}}$. By T_2, let E be transformed into $\bar{E}$ and let the complex B be transformed into $\bar{B}$. Then B^2 does not necessarily transform into $\bar{B}^2$, and similarly for the higher powers of B. In fact if we consider the multiplication table of E in the square array indicated schematically thus:

	B	B^2	B^3	$\ldots$
B	B^2	B^3	B^4	$\ldots$
B^2	B^3	B^4	$\ldots$	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

,

then T_2 adds to each element in the j-th horizontal strip and the k-th vertical

strip, linear combinations of elements in the j-th horizontal strip and below the j-th, and in the vertical strips in the k-th and to the right of the k-th. Hence the statement at the beginning of this section.

Let n' be the order of B, and in general let $n^{(i)}$ be the order of $B_i \equiv B^i (\text{mod } E^{i+1})$. Then we can assume that $E = (e_1, \ldots, e_n)$ is in such a form that $B_1 = B = (e_1, \ldots, e_{\nu'})$, $B_2 = (e_{\nu'+1}, \ldots, e_{\nu''})$, and in general $B_i = (e_{\nu^{(i-1)}+1}, \ldots, e_{\nu^{(i)}})$ for $i \leq \alpha - 1$, where $\nu^{(j)} = \sum_{k=1}^{j} n^{(k)}$. Now, $n^{(\alpha-1)} \geq 1$, $n = \sum_{i=1}^{\alpha-1} n^{(i)}$ and $E = \sum_{i=1}^{\alpha-1} B_i$. We shall henceforth make this assumption for algebras in the canonical form. The group G of all linear transformations which leave unaltered the canonical form (under this slightly modified definition) is by § 4 seen to be the group of all non-singular transformations which, for every i, replace an element of B_i by the sum of an element of B_i and an element of E^{i+1}.

6. *Definition of Invariants of Nilpotent Algebras.* Given fixed positive integers n and $\mu^0 = 0, \mu', \ldots, \mu^{(\alpha-1)}$ such that $\sum_i \mu^{(i)} = n$, let $A = (e_1, \ldots, e_n)$ be an algebra over the field F such that, if we denote the complex $(e_{\mu^{(i-1)}+1}, \ldots, e_{\mu^{(i)}})$ by C_i $(i = 1, \ldots, \alpha - 1)$, then the product of two complexes C_j and C_k will be expressible linearly and homogeneously in terms of the elements of C_l, where $l \geq j + k$, and where in particular $C_j C_k = 0$ when $j + k \geq \alpha$; but otherwise let the constants of multiplication of the algebra A be undetermined elements of F. A is necessarily nilpotent; and furthermore, the particular algebras $A_1, A_2, \ldots$ obtained from A by assigning to the constants of multiplication particular sets of values in the field F include all those n-ary nilpotent algebras of index α where, for every i, $n^{(i)} = \mu^{(i)} - \mu^{(i-1)}$ is the order of the complex B_i defined in § 5. For convenience, we will say such algebras are of *genus* $(\alpha; n', \ldots, n^{(\alpha-1)})$.

Moreover, if G is the group which leaves unaltered the canonical form of all algebras of genus $(\alpha; n', \ldots, n^{(\alpha-1)})$, then any transformation of the group G will carry any particular one of the algebras $A_1, A_2, \ldots$ into another one of the set, and in particular it will carry the set of all algebras of genus $(\alpha; n', \ldots, n^{(\alpha-1)})$ into itself. Accordingly the particular algebras $A_1, A_2, \ldots$ may be separated into *classes* $\mathfrak{C}_1, \ldots, \mathfrak{C}_\sigma$ such that two of the algebras belong to the same class if and only if they be equivalent with respect to the field F under the group G; that is, if and only if there is a transformation of the group G with coefficients in F which will carry one algebra into the other.

A single-valued function $\mathfrak{J}$ of the constants of multiplication of the algebras A is called an *invariant* of A under the group G if, for $j = 1, \ldots, \sigma$, this func-

tion has the same value v_j for all algebras in the class C_j. A set of invariants $\mathfrak{J}_1, \ldots, \mathfrak{J}_\rho$ is said to *completely characterize* the algebras of genus $(\alpha; n', \ldots, n^{(\alpha-1)})$ over the field F under the group G when each $\mathfrak{J}_k$ has the same value for two algebras of the genus if and only if they belong to the same class. The *characteristic invariant* I_k for the class $\mathfrak{C}_k$ is defined to be that invariant which has the value unity for algebras of the class $\mathfrak{C}_k$ and the value zero for all other algebras of the same genus.

In order to find invariants in the sense defined above, we will also consider rational (absolute and relative) invariants and covariants for the algebras of a given genus.

7. *Importance of the Complex B for Associative Algebras.* For an associative algebra E, the complex $B \equiv E \pmod{E^2}$ in some manner determines the behavior of the whole algebra. If also B is commutative, so is E.

On the contrary, if B is such that, when e_1 and e_2 are any two elements of B, $e_1 e_2 = -e_2 e_1$, then by induction it follows that B^i is commutative with B^j unless i and j are both odd; and if i and j are both odd, we have $e^{(i)} e^{(j)} = -e^{(j)} e^{(i)}$, where $e^{(k)}$ is any element in B^k.

Moreover, an associative nilpotent algebra E is reducible if and only if B is reducible.* First, if B is reducible, $B = B_1 + B_2$, where $B_1 B_2 = 0 = B_2 B_1$ and $B_1 \cap B_2 = 0$. Therefore, in view of the associative law, $B^i = B_1^i + B_2^i$; and accordingly, if we take $E_1 = \sum_{i=1}^{\alpha-1} B_1^i$ and $E_2 = \sum_{i=1}^{\alpha-1} B_2^i$ (where α is the index of E), then $E = E_1 + E_2$, with $E_1 E_2 = 0 = E_2 E_1$, where E_1 and E_2 are algebras.

Conversely, if E is reducible, with $E = E_1 + E_2$, where $E_1 \cap E_2 = 0$, then $E_1 = \sum_{i=1}^{\alpha-1} B_1^i$ and $E_2 = \sum_{i=1}^{\alpha-1} B_2^i$, where $B_1 \equiv E_1 \pmod{E_1^2}$ and $B_2 \equiv E_2 \pmod{E_2^2}$. Therefore, $B_1 \cap B_2 = 0$, and $B_1 B_2 = 0 = B_2 B_1$, $B_1 + B_2 = B$.

8. *"Special" Canonical Form.* As we stated at the beginning of § 5, several of the complexes $B, \ldots, B^{\alpha-1}$ may overlap, but these "overlappings" are more or less restricted in view of one or two simple properties which follow from the formulæ $E^p = \sum_{k=p}^{\alpha-1} B^k$ in § 3. In an associative algebra, for $a < \alpha$, $b < \alpha$, $B^a = B^b$ if and only if $a = b$; more generally, B^{a+b} does not contain B^a, where $a < \alpha$ and $b > 0$. Further, we can not have $B^c = B^a + B^b$ for $a, b < \alpha$ unless $a = c$ or $b = c$. Finally, for every exponent $a < \alpha$, there are numbers in B^a which can not be expressed as linear homogeneous functions of elements in $B^{a+1}, \ldots, B^{\alpha-1}$.

Now, if an algebra in canonical form be such that $B^i \cap B^j = 0$ $(i \neq j;\ i, j < \alpha)$, it will be said to be in *"special" canonical form.* In particular, in view of the

* Wedderburn, *loc. cit.*, p. 88.

properties mentioned above, all associative algebras in one, two, three or four units are necessarily in "special" canonical form, except for two types of quaternary algebras, namely $n'=2$, $n''=n^{(3)}=1$ and $n'=n''=n^{(3)}=n^{(4)}=1$.

THEOREM. *If G' be the group of all linear transformations with coefficients in F which leaves unaltered the "special" canonical form of the nilpotent algebra $E=B+E^2$ over the field F, and if G'' be that subgroup of G' which transforms the units of B among themselves alone, the units of B^2 among themselves alone, etc., then the invariants of E under the group G'' are all the invariants under G'. Furthermore, if two nilpotent algebras in "special" canonical form over a field F are equivalent with respect to F, they are equivalent under a transformation of the group G''.*

To show this, let T' be any transformation of the group G'. Then $T'=T_1'T''$, where T'' is in G'' and where, for every i, T_1' at most adds to each unit of B^i an element in E^{i+1}. T_1' is in G', and accordingly, from its nature and in view of the definition of "special" canonical form, can not affect the constants of multiplication. Hence, the theorem.

Invariants for General Nilpotent Algebras in Canonical Form, §§ 9–12.

9. *Two Classical Methods Which Furnish no Invariants.* Since the characteristic determinants for an n-ary nilpotent algebra are $\delta(\omega)\equiv\omega^n\equiv\delta'(\omega)$, all invariants of nilpotent algebras which can be obtained by means of the theorem of § 1 are zero.

Furthermore, no information in regard to nilpotent algebras is furnished by Scheffers' theorem that, for an associative algebra with a modulus, if the right-hand characteristic equation $\delta(\omega)=0$ defines ω as an h-valued function of the coordinates of the general number of the algebra, then we may choose as normalized units $\varepsilon_1, \ldots, \varepsilon_h, \eta_1, \ldots, \eta_k$ satisfying certain conditions. For, if we adjoin a modulus e_0 to the nilpotent algebra $(e_1, \ldots, e_n)$, the right-hand characteristic equation $(x_0-\omega)^{n+1}=0$ of the resulting algebra defines ω as a single-valued function of the x's.

10. *Invariants Obtained from the Parastrophic Matrix.* The argument of § 5 shows that certain matrices obtained from the parastrophic matrix are unaltered under transformations of the group G.

If we have an algebra $E=(e_1, \ldots, e_n)$, not necessarily nilpotent, over the field F with multiplication table (1), the *parastrophic matrix** is defined as

$$R\equiv\left(\sum_{k=1}^{n}\gamma_{ijk}\xi_k\right), \tag{4}$$

* This term and the notation are those used by Frobenius, *Berliner Sitzungsberichte,* 1903, p. 522. The covariance of R is also due to him (*loc. cit.*).

where the ξ_k are variables ranging independently over F which are cogredient with the e_k. If we subject the units of E to the non-singular transformation

$$\bar{e}_i = \sum_{j=1}^{n} a_{ij}\, e_j \quad (i = 1, \ldots, n),$$

where the a's are in F, then the parastrophic matrix $\bar{R}$ for the new algebra $\bar{E}$ is

$$\bar{R} = A\, R\, A', \tag{5}$$

where the prime indicates the conjugate matrix. Thus R is a covariant of the algebra.

In particular, let $E = \sum_{i=1}^{\alpha-1} B_i$ be a nilpotent algebra of genus

$$(\alpha;\, n', \ldots, n^{(\alpha-1)})$$

in canonical form. Then the parastrophic matrix is

$$R \equiv \left|\begin{array}{c|c|c} \begin{array}{c}\displaystyle\sum_{r=\nu'+1}^{n} \gamma_{pqr}\, \xi_r \\ (p, q = 1, \ldots, \nu')\end{array} & \begin{array}{c}\displaystyle\sum_{\nu''+1}^{n} \gamma_{pqr}\, \xi_r \\ \begin{pmatrix} p = 1, \ldots, \nu' \\ q = \nu' + 1, \ldots, \nu'' \end{pmatrix}\end{array} & \cdots\cdots\cdots \\ \hline \begin{array}{c}\displaystyle\sum_{\nu''+1}^{n} \gamma_{pqr}\, \xi_r \\ \begin{pmatrix} p = \nu' + 1, \ldots, \nu'' \\ q = 1, \ldots, \nu' \end{pmatrix}\end{array} & \begin{array}{c}\displaystyle\sum_{\nu^{(3)}+1}^{n} \gamma_{pqr}\, \xi_r \\ (p, q = \nu'+1, \ldots, \nu'')\end{array} & \cdots\cdots\cdots \\ \hline \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots \end{array}\right|, \tag{6}$$

where $\nu^{(i)} = \sum_{j=1}^{i} n^{(j)}$ $(i = 1, \ldots, \alpha - 1)$.

If we subject the algebra to a transformation T of the group G which leaves unaltered the canonical form, then $T = T_2\, T_1$, where T_1 transforms the units of B_1 among themselves alone, the units of B_2 among themselves alone, etc.; and where, for every i, T_2 at most adds to each unit of B_i an element in E^{i+1}. Under the transformation T_1, R is subjected to a finite number of elementary transformations in such a way that any "box" of the schematic matrix (6) is subjected to a finite number of elementary transformations; and at the same time the variables ξ_k which occur in any partial sum

$$\sum_{r=\nu^{(i-1)}+1}^{\nu^{(i)}} \gamma_{pqr}\, \xi_r \tag{7}$$

are subjected to a non-singular transformation among themselves alone. By

the transformation T_2, R is subjected to a finite number of elementary transformations in such a way that to each row of the "box" in the j-th horizontal strip and k-th vertical strip of the schematic matrix are added linear combinations of the rows from "boxes" which are in the k-th vertical strip below the j-th horizontal strip, and a similar transformation is then made on the columns. At the same time, by T_2 the variables ξ which occur in any partial sum (7) have added to them linear combinations of the variables ξ with a larger subscript. In short, under the transformation T, each partial sum (7) in the transformed matrix is affected only by those partial sums (7) in the original matrix having r range from $\nu'+1$ to $\nu^{(i)}$ which are in the same horizontal strip or in the horizontal strips below, and at the same time in the same vertical strip or in the vertical strips to the right.

That is, for every value of $0 \leq l \leq \alpha-1$, the square matrix obtained from R by erasing the first l horizontal strips and the first l vertical strips of the schematic matrix (6) is subjected to a finite number of elementary transformations under the group G. Also, if for any m we delete the last m horizontal strips and the last m vertical strips of the schematic matrix, and if at the same time we erase all partial sums (7) where $i \geq m$, then the resulting matrix is subjected to a finite number of elementary transformations under the group G.

Now just as any homogeneous algebraic covariant (or invariant) of a covariant of a quantic is itself a covariant (or invariant) of the quantic, so any homogeneous algebraic covariant (or invariant) of a function of the constants of multiplication and coordinates which is invariantive under the group G for algebras of genus $(\alpha;\, n', \ldots, n^{(\alpha-1)})$ is itself a covariant (or invariant) under the group G for these algebras. Hence, every homogeneous algebraic covariant (or invariant) and every arithmetic invariant of a matrix obtained from (6) by one of the two methods of deletion described above is in turn a covariant (or invariant) under the group G for the n-ary algebras of genus $(\alpha;\, n', \ldots, n^{(\alpha-1)})$.

Finally, for every l and m, the matrix obtained from (6) *by combining the two methods of deletion described above is a covariant under the group G for the algebras of genus* $(\alpha;\, n', \ldots, n^{(\alpha-1)})$, *and this matrix furnishes further covariants and invariants for such algebras under the group G.*

Hence, if two nilpotent algebras E' and E'' of index α are equivalent, then if we delete E' by $E'^{\alpha-m}$ and E'' by $E''^{\alpha-m}$, the resulting algebras are equivalent, and this is true for every positive integer $m<\alpha$. Similarly for every positive integer $l<\alpha$, if we erase in E' all units in $B', \ldots, B'_l$ and at

the same time erase in E'' all units in $B'', \ldots, B''_l$, where $B' \equiv E' \pmod{E'^2}$ and $B'' \equiv E'' \pmod{E''^2}$, then the resulting algebras are equivalent.

Moreover, there are other invariants of these matrices which are a generalization of the rational invariant factors of a matrix whose elements are polynomials in one variable. Consider a particular matrix M obtained from R by one of the above methods of deletion, and let it be of rank r if and only if the ξ_k take on a set of values linearly dependent on the t_r linearly independent sets $\xi_k = \xi_k^{(r,s)}$ $(s=1, \ldots, t_r)$, where k has the same range as in M. The numbers t_r $(r=1, \ldots)$ are unaltered under a transformation of the group G.

11. *Invariants of the Complexes B_i. Fundamental Quadratics, Cubics, etc.* Consider the complex $B_i = (e_{\nu(i-1)+1}, \ldots, e_{\nu(i)})$ of the algebra E of index α, over the field F (see § 5). Modulo E^{2i+1}, the square of the general number

$$X^{(i)} = \sum_{p=\nu(i-1)+1}^{\nu(i)} x_p e_p \tag{8}$$

of B_i is

$$X^{(i)2} \equiv \sum_{p,\, q=\nu(i-1)+1}^{\nu(i)} \sum_{r=\nu(2i-1)+1}^{\nu(2i)} \gamma_{pqr} x_p x_q e_r \pmod{E^{2i+1}}. \tag{9}$$

Define

$$\mathcal{Q}^{(i)} = \sum_{r=\nu(2i-1)+1}^{\nu(2i)} \mathcal{Q}_r^{(i)} \eta_r, \tag{10}$$

where

$$\mathcal{Q}_r^{(i)} \equiv \sum_{p,\, q=\nu(i-1)+1}^{\nu(i)} \gamma_{pqr} x_p x_q \quad (r = \nu^{(2i-1)}+1, \ldots, \nu^{(2i)}), \tag{11}$$

and where the η's are variables ranging independently over F and such that, when the units of B_{2i} are subjected to the transformation

$$\left.\begin{aligned} e'_p &= \sum_{q=\nu(2i-1)+1}^{\nu(2i)} a_{pq} e_q + \sum_{q>\nu(2i)} a_{pq} e_q \quad (\nu^{(2i-1)}+1 \leq p \leq \nu^{(2i)}), \\ &| a_{pq} | \neq 0 \quad (p, q = \nu^{(2i-1)}+1, \ldots, \nu^{(2i)}), \end{aligned}\right\} \tag{12}$$

then the η's are subjected to the transformation

$$\eta'_p = \sum_{q=\nu(2i-1)+1}^{\nu(2i)} a_{pq} \eta_q \quad (\nu^{(2i-1)}+1 \leq p \leq \nu^{(2i)}). \tag{13}$$

The necessary and sufficient condition that there exist in B_i τ linearly independent numbers such that their squares are all zero (modulo E^{2i+1}) is that the $n^{(2i)}$ homogeneous quadratics (11) should have in common τ linearly independent solutions. Hence the number of linearly independent solutions common to the $\mathcal{Q}_r^{(i)}$ has the invariantive property, and the resultant of the $\mathcal{Q}_r^{(i)}$ is an invariant.

If we subject the units of $E = \sum_{i=1}^{\alpha-1} B_i$ to a non-singular transformation T of

the group G, the units in B_{2i} are subjected to a transformation of the form (12) and the η's to the corresponding transformation (13). Now $T = T_1 \ldots . T_{\alpha-1}$, where T_j leaves unaltered all units except those in B_j, and replaces each unit in B_j by the sum of a number in B_j and a number in E^{j+1}. The quadratics (11) are affected at most by T_i and T_{2i}.

If e_p and e_q be any two units in B_i, and if γ'_{pqr} are the constants of multiplication in the algebra obtained by applying to E the transformation T_{2i}, then by the argument just after (7) we must have

$$\sum_{r=\nu^{(2i-1)}+1}^{\nu^{(2i)}} \gamma_{pqr}\eta_r = \sum_{r=\nu^{(2i-1)}+1}^{\nu^{(2i)}} \gamma'_{pqr}\eta'_r,$$

holding identically in view of the relations between the η's and the η''s. Hence

$$\mathcal{Q}^{(i)}(\gamma_{pqr};\ x_p;\ \eta_r) \equiv \mathcal{Q}^{(i)}(\gamma'_{pqr};\ x'_p;\ \eta'_r)$$

under the transformation T_{2i}.

Finally, if γ''_{pqr} be the constants of multiplication in the algebra obtained by applying to E the transformation T_i, then since $e_{\nu^{(2i-1)}+1}, \ldots, e_{\nu^{(2i)}}$ are unaltered by T_i, the square of $X^{(i)} = \Sigma x''_p e''_p \pmod{E^{2i+1}}$ can be derived from the square of $X^{(i)} = \Sigma x_p e_p \pmod{E^{2i+1}}$ by replacing γ_{pqr} by γ''_{pqr} and x_p by x''_p; and accordingly

$$\mathcal{Q}^{(i)}_r(\gamma_{pqr};\ x_p) \equiv \mathcal{Q}^{(i)}_r(\gamma''_{pqr};\ x''_p) \quad (\nu^{(2i-1)}+1 \leq r \leq \nu^{(2i)})$$

in view of the relations between the x_p and the x''_p. Thus $\mathcal{Q}^{(i)}$ is unaltered under T_i.

Combining these results, we see that the quadratic form (10) is unaltered under transformations of the group G, and we shall refer to $\mathcal{Q}^{(i)}$ as a covariant of the complex B_i; in particular we shall call it the *fundamental quadratic for* B_i. Moreover, the discriminant of (10) when we regard the x's as variables and the η's as parameters is a function which has the invariantive property for the general algebra of genus $(\alpha;\ n', \ldots, n^{(\alpha-1)})$. Also the invariants and covariants of this discriminant regarded as a function of the η's are invariants and covariants for the algebras of genus $(\alpha;\ n', \ldots, n^{(\alpha-1)})$.

Similarly the cubic form

$$\mathcal{C}^{(i)} \equiv \sum_{r=\nu^{(3i-1)}+1}^{\nu^{(3i)}} \mathcal{C}^{(i)}_r \eta_r, \tag{14}$$

where

$$\mathcal{C}^{(i)}_r \equiv \sum_{j,k,l} \Big(\sum_m \gamma_{klm}\gamma_{jmr}\Big) x_j x_k x_l \quad \begin{pmatrix} \nu^{(i-1)}+1 \leq j,\ k,\ l \leq \nu^{(i)} \\ \nu^{(2i-1)}+1 \leq m \leq \nu^{(2i)} \end{pmatrix},$$

is a covariant for nilpotent algebras of genus $(\alpha;\ n', \ldots, n^{(\alpha-1)})$. Here the η's are related to the units in B_{3i} as the η's of (10) were related to the units in B_{2i}. In an analogous manner we can form a covariantive quartic, quintic,

etc., for the complex B_i, and continue until we get a p-ic form which is identically zero in the parameters η and the variables x for all γ's. For brevity we shall call these the *fundamental forms for the complex* B_i, although they depend also on the complexes B_{2i}, B_{3i}, etc. In particular, for the complex $B \equiv E (\mathrm{mod}\ E^2)$ there are $\alpha - 2$ fundamental forms not identically zero for the algebra of index α.

CHAPTER III.

CLASSIFICATION OF GENERAL NILPOTENT ALGEBRAS WITH $n \leq 4$.

By using the theory of the previous chapter, we can readily classify the nilpotent algebras (not necessarily associative) in a small number of units. We give below the canonical forms of the multiplication tables of all nilpotent algebras over the field C of ordinary complex numbers, having four units or less, to which any such algebra is equivalent under a non-singular transformation with coefficients in C.* Furthermore, no two of the algebras tabulated are equivalent. Where they are not too awkward, we give also the invariantive conditions that a given algebra in canonical form be equivalent to one of the algebras tabulated. Throughout, we shall use α to denote the index of the algebra E, and $n^{(i)}$ for the number of linearly independent elements in $B_i \equiv B^i (\mathrm{mod}\ E^{i+1})$, where $B \equiv E$ (mod E^2). For convenience, we shall understand that products not written are zero.

12. $n = 1, 2$.

$$n = 1: \quad e_1 e_1 = 0.$$
$$n = 2: \quad \alpha = 2, \quad e_i e_j = 0 \quad (i, j = 1, 2).$$
$$\alpha = 3, \quad e_1^2 = e_2.$$

13. $n = 3$.

Type A: $n' = 3$, $n'' = n^{(3)} = 0$; $\alpha = 2$, $e_i e_j = 0 \quad (i, j = 1, 2, 3)$.

Type B: $n' = 2$, $n'' = 1$, $n^{(3)} = 0$; $\alpha = 3$.

$$Q_3 \equiv x_1^2 \gamma_{113} + x_1 x_2 (\gamma_{123} + \gamma_{213}) + x_2^2 \gamma_{223}; \; D_3 \equiv (\gamma_{123} + \gamma_{213})^2 - 4\gamma_{113}\gamma_{223}.$$

I. $Q_3 \equiv 0$: $e_1 e_2 = -e_2 e_1 = e_3$.

II. $Q_3 \not\equiv 0$; $D_3 \neq 0$: $e_2 e_1 = e_3$, $e_1 e_2 = \lambda e_3$, $\lambda \neq -1$, $|\lambda| \leq 1$.

III. $Q_3 \not\equiv 0$; $D_3 = 0$: $e_1 e_1 = e_3$; $e_1 e_1 = e_3 = e_1 e_2 = -e_2 e_1$.

The classes of Type B (in canonical form) are completely characterized by the ranks of

$$\begin{vmatrix} \gamma_{113} & \dfrac{\gamma_{123} + \gamma_{213}}{2} \\ \dfrac{\gamma_{123} + \gamma_{123}}{2} & \gamma_{223} \end{vmatrix}, \quad \begin{vmatrix} \gamma_{113} & \gamma_{123} \\ \gamma_{213} & \gamma_{223} \end{vmatrix}$$

and the absolute invariant $(\gamma_{113}\gamma_{223} - \gamma_{123}\gamma_{213})/D_3$.

*Most of the work, however, holds for any field F. These results have been compared and (as far as possible) have been checked with Allen's results for associative nilpotent algebras in 2, 3, 4 units over any field F (*Transactions*, Vol. IX, pp. 203–218).

Type C: $n' = n'' = n^{(3)} = 1$; $\alpha = 4$,

$$e_1 e_1 = e_2,\ e_1 e_2 = e_3,\ e_2 e_1 = \lambda e_3;\ e_1 e_1 = e_2,\ e_2 e_1 = e_3.$$

Here there is a single infinitude of classes characterized among themselves by the values of $r_{1,\lambda}$ and $r_{0,1}$, the ranks of

$$\begin{vmatrix} \gamma_{123} & 1 \\ \gamma_{213} & \lambda \end{vmatrix}, \quad \begin{vmatrix} \gamma_{123} & 0 \\ \gamma_{213} & 1 \end{vmatrix}$$

respectively.

Type D: $n' = n^{(2)} = 1$, $n^{(3)} = 0$, $n^{(4)} = 1$,

$$e_1 e_1 = e_2,\ e_2 e_1 = \varkappa e_3,\ e_2 e_2 = e_3.$$

14. $n = 4$.

Type A: $n' = 4$, $n'' = n^{(3)} = n^{(4)} = 0$; $\alpha = 2$, $e_i e_j = 0$ $(i, j = 1, \ldots, 4)$.

Type B: $n' = 3$, $n'' = 1$, $n^{(3)} = n^{(4)} = 0$; $\alpha = 3$.

Let r be the rank of D_4, the discriminant of the covariantive quadratic $Q_4 \equiv \sum_{i,j}^{1,2,3} \gamma_{ij4} x_i x_j$; and let ρ be the rank of $R = |\gamma_{ij4}|$ $(i, j = 1, 2, 3)$.

I. $r = 0$, $e_1 e_2 = -e_2 e_1 = e_4$.

II. $r = 1$, (a) $\rho = 3$, $e_1^2 = e_2 e_3 = -e_3 e_2 = e_4$.

(b) $\rho = 2$, $e_1^2 = e_1 e_2 = -e_2 e_1 = e_4$.

(c) $\rho = 1$, $e_1^2 = e_4$.

III. $r = 2$, (a) $\rho = 3$, $e_2 e_1 = e_1 e_3 = -e_3 e_1 = e_2 e_3 = -e_3 e_2 = e_4$.

(b) $\rho = 2$, $e_2 e_1 = e_1 e_3 = -e_3 e_1 = e_4$.

(c_γ) $\rho = 1$, $e_1 e_2 = e_4$, $e_2 e_1 = \gamma e_4$, $|\gamma| \leq 1$.

The algebras of III (c) are characterized completely among themselves as follows: Let (x_{i1}, x_{i2}, x_{i3}) $(i = 1, 2, 3)$ be three linearly independent solutions of $Q_4 = 0$, and let (x_{31}, x_{32}, x_{33}) be the double solution. Then the class (c_γ) is characterized by the fact that

$$\begin{vmatrix} 1 & \sum_{k,l}^{1,2,3} x_{1k} x_{2l} \gamma_{kl4} \\ \gamma & \sum_{k,l}^{1,2,3} x_{2k} x_{1l} \gamma_{kl4} \end{vmatrix} \times \begin{vmatrix} 1 & \sum_{k,l}^{1,2,3} x_{2l} x_{1k} \gamma_{kl4} \\ \gamma & \sum_{k,l}^{1,2,3} x_{1k} x_{2l} \gamma_{kl4} \end{vmatrix} = 0.$$

IV. $r = 3$, (a) $\rho = 2$, $-e_2 e_1 = e_3 e_1 = e_2 e_3 = e_3 e_2 = e_4$.

(b_γ) $\rho = 3$, $e_2 e_3 = e_3 e_2 = e_1^2 = e_2 e_1 = e_4$, $e_3 e_1 = \gamma e_4$.

The classes of B, IV are completely characterized among themselves by r, ρ and the absolute invariant R/D_4.

Type C: $n' = 2$, $n'' = 2$, $n^{(3)} = n^{(4)} = 0$; $\alpha = 3$. Here the fundamental quadratic is $Q \equiv Q_3 \eta_3 + Q_4 \eta_4$, where $Q_k \equiv \sum_{i,j}^{1,2} \gamma_{ijk} x_i x_j$ $(k = 3, 4)$. Let ρ be the

rank of the discriminant of $\left|\sum_k^{3,4} \gamma_{ijk}\eta_k\right|$ $(i, j = 1, 2)$ for general η_3, η_4; and let $\Theta = \frac{\vartheta(\vartheta-1)}{2}$, where ϑ is the rank of

$$\begin{pmatrix} \gamma_{113} & \gamma_{123}+\gamma_{213} & \gamma_{223} \\ \gamma_{114} & \gamma_{124}+\gamma_{214} & \gamma_{224} \end{pmatrix};$$

and let R be the resultant of Q_3 and Q_4.

I. $\Theta = 0$, $R = 0$, $e_1e_2 = e_3$, $e_2e_1 = e_4$.

II. $\Theta = 1$, $R = 0$, (a) $\rho = 0$, $e_1^2 = e_4$, $e_1e_2 = e_3$; $e_1^2 = e_4$, $e_2e_1 = e_3$.

(b_k) $\rho = 1$, $e_1^2 = e_4$, $e_2e_1 = e_3$, $e_1e_2 = ke_3$ $(k \neq -1, 0)$.

(c) $\rho = 2$, $e_1^2 = e_1e_2 = e_4$, $e_2e_1 = e_3$.

The classes in Case II are characterized completely among themselves by $r_{k,1}$ and $r_{1,0}$, the ranks of

$$\begin{pmatrix} A_1 & A_2 & k \\ B_1 & B_2 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} A_1 & A_2 & 1 \\ B_1 & B_2 & 0 \end{pmatrix}$$

respectively, where

$$A_1 = x_1[\gamma_{113} + (x_2-1)\gamma_{213}] + x_2[\gamma_{123} + (x_2-1)\gamma_{223}],$$
$$A_2 = x_1[\gamma_{114} + (x_2-1)\gamma_{214}] + x_2[\gamma_{124} + (x_2-1)\gamma_{224}],$$
$$B_1 = x_1[\gamma_{113} + (x_2-1)\gamma_{123}] + x_2[\gamma_{213} + (x_2-1)\gamma_{223}],$$
$$B_2 = x_1[\gamma_{114} + (x_2-1)\gamma_{124}] + x_2[\gamma_{214} + (x_2-1)\gamma_{224}],$$

where (x_1, x_2) is the solution $\neq (0, 0)$ common to $Q_3 = 0$ and $Q_4 = 0$.

III. $\Theta = 1$, $R \neq 0$.

(a_λ) $e_1^2 = e_2^2 = e_4$, $e_2e_1 = e_3$, $e_1e_2 = e_3 + \lambda e_4$.

There is a class for every value of λ^2.

Type D: $n' = 2$, $n'' = 1$, $n^{(3)} = 1$; $\alpha = 4$. Since $\left|\frac{\gamma_{ij3}+\gamma_{ji3}}{2}\right|$ $(i, j = 1, 2)$ is invariantive for classes of this type, we can make use of the results for Type B of the ternary algebras. Let r be the rank of $Q_3 \equiv \sum_{i,j}^{1,2} \gamma_{ij3}x_ix_j$, ρ the rank of

$$\begin{vmatrix} \gamma_{134} & \gamma_{314} \\ \gamma_{234} & \gamma_{324} \end{vmatrix}$$

I. $r = 0$.

(a) $\rho = 1$.

(1_k) $e_1e_2 = -e_2e_1 = e_3$, $e_1e_3 = e_2e_2 = e_4$, $e_3e_1 = ke_4$ $(k \neq -1)$.

(2_k) $e_1e_2 = -e_2e_1 = e_3$, $e_1e_3 = e_4$, $e_3e_1 = ke_4$.

(3) $e_1e_1 = e_1e_3 = -e_3e_1 = e_2e_2 = e_4$, $e_1e_2 = -e_2e_1 = e_3$.

(4) $e_1 e_2 = -e_2 e_1 = e_3$, $e_1 e_3 = -e_3 e_1 = e_2 e_2 = e_4$.

(5) $e_1 e_3 = -e_3 e_1 = e_4$, $e_1 e_2 = -e_2 e_1 = e_3$, $e_1 e_1 = e_4$.

(6) $e_1 e_3 = -e_3 e_1 = e_4$, $e_1 e_2 = -e_2 e_1 = e_3$.

(7) $e_1 e_3 = -e_3 e_1 = e_4$, $e_1 e_2 = e_3$, $e_2 e_1 = -e_3 + e_4$.

The classes of Type D, I (a) are characterized completely among themselves by the vanishing or non-vanishing of

$$\gamma_{134}^2 \gamma_{224} - \gamma_{134} \gamma_{234} (\gamma_{124} + \gamma_{214}) + \gamma_{234}^2 \gamma_{114},$$

and by the value of $r_{1,k}$, $r_{0,1}$ and $(2 - r_{1,-1}) \rho_1$, where $r_{1,k}$, $r_{0,1}$ and ρ_1 are respectively the ranks of

$$\begin{pmatrix} \gamma_{134} & \gamma_{234} & 1 \\ \gamma_{314} & \gamma_{324} & k \end{pmatrix}, \quad \begin{pmatrix} \gamma_{134} & \gamma_{234} & 0 \\ \gamma_{314} & \gamma_{324} & 1 \end{pmatrix}, \quad \begin{pmatrix} 2\gamma_{114} & \gamma_{124} + \gamma_{214} \\ \gamma_{124} + \gamma_{214} & 2\gamma_{224} \end{pmatrix}.$$

(b). $\rho = 2$.

(1) $e_1 e_2 = -e_2 e_1 = e_3$, $e_1 e_3 = e_3 e_2 = e_4$.

(2) $e_1 e_2 = -e_2 e_1 = e_3$, $e_1 e_3 = e_3 e_2 = e_1 e_1 = e_4$.

These two classes are distinguished by the vanishing and non-vanishing respectively of

$$\gamma_{114} (\gamma_{234} + \gamma_{324})^2 - (\gamma_{124} + \gamma_{214}) (\gamma_{234} + \gamma_{324}) (\gamma_{134} + \gamma_{314}) + \gamma_{224} (\gamma_{134} + \gamma_{314})^2.$$

II. $r = 1$.

(a) $\gamma_{123} + \gamma_{213} \neq 0$.

(1) $\mathfrak{M} \equiv \left| \sum_{k}^{3,4,5} \gamma_{ijk} \xi_k \right| \equiv 0 \ (i, j = 1, 2, 3)$. ∞^2 classes.

(a′) $e_1 e_1 = e_1 e_2 = e_3$, $e_2 e_1 = -e_3 + a e_4$, $e_1 e_3 = e_4$, $e_3 e_1 = k e_4$ ($k \neq 0$; $a = 0, 1$).

(b′) $e_1 e_1 = e_3 = e_1 e_2 = -e_2 e_1$, $e_2 e_2 = a e_4$, $e_1 e_3 = e_4$ ($a = 0, 1$).

(c′) $e_1 e_1 = -e_2 e_1 = e_3$, $e_1 e_2 = e_3 + a e_4$, $e_2 e_3 = e_4$.

(d′) $e_1 e_1 = e_1 e_2 = -e_2 e_1 = e_3$, $e_3 e_1 = e_4$.

(e′) $e_1 e_1 = e_1 e_2 = -e_2 e_1 = e_3$, $e_2 e_2 = e_3 e_1 = e_4$.

(f′) $e_1 e_1 = e_1 e_2 = e_3$, $e_2 e_1 = -e_3 + a e_4$, $e_3 e_2 = e_4$ ($a = 0, 1$).

(g′) $e_1 e_1 = e_1 e_2 = e_3$, $e_2 e_1 = -e_3 + a e_4$, $e_2 e_3 = -e_3 e_2 = e_4$, $e_3 e_1 = \lambda e_4$ ($a = 0, 1$).

(h′) $e_1 e_1 = e_1 e_2 = e_3$, $e_2 e_1 = -e_3 + k e_4$, $e_2 e_2 = e_2 e_3 = -e_3 e_2 = e_4$, $e_3 e_1 = \lambda e_4$.

(i′) $e_1 e_1 = e_1 e_2 = -e_2 e_1 = e_3$, $e_3 e_1 = e_3 e_2 = \lambda e_4$, $e_2 e_3 = e_4$ ($\lambda \neq -1$).

(j′) $e_1 e_1 = e_1 e_2 = e_3$, $e_2 e_1 = -e_3 + k e_4$, $e_3 e_1 = e_3 e_2 = \lambda e_4$, $e_2 e_3 = e_4$ ($k = 0, 1$; $\lambda \neq -1$).

(2) $\mathfrak{M}$ a perfect cube not identically zero. ∞^2 classes.

(a′) $e_1 e_1 = e_1 e_2 = -e_2 e_1 = e_3$, $e_2 e_2 = e_1 e_3 = e_4$, $e_3 e_1 = \lambda e_4$.

(b′) $e_1 e_1 = e_3$, $e_2 e_3 = -e_3 e_2 = \lambda e_4$, $e_2 e_2 = e_4$, $e_1 e_2 = -e_2 e_1 = e_3 + k e_4$, $e_3 e_1 = e_4$.

(c′) $e_1 e_1 = e_3$, $e_1 e_2 = -e_2 e_1 = e_3 + e_4$, $e_2 e_3 = -e_3 e_2 = \lambda e_4$, $e_3 e_1 = e_4$.

(d′) $e_1 e_1 = e_1 e_2 = -e_2 e_1 = e_3$, $e_3 e_1 = e_2 e_3 = e_4$, $e_3 e_2 = k e_4$.

(e′) $e_1 e_1 = e_3$, $e_1 e_2 = -e_2 e_1 = e_3 + e_4$, $e_3 e_1 = e_2 e_3 = e_4$, $e_3 e_2 = k e_4$.

(3) $\mathfrak{M}$ not a perfect cube. ∞^2 classes.

(a′) $e_1 e_1 = -e_2 e_1 = e_3$, $e_1 e_2 = e_3 + a e_4$, $e_2 e_3 = -e_3 e_2 = e_4$, $e_3 e_1 = \lambda e_4$ $(a = 0, 1)$.

(b′) $e_1 e_1 = -e_2 e_1 = e_3$, $e_1 e_2 = e_3 + a e_4$, $e_2 e_2 = e_2 e_3 = -e_3 e_2 = e_4$, $e_3 e_1 = \lambda e_4$ $(a = 0, 1;\ \lambda \neq 0)$.

(c′) $e_1 e_1 = -e_2 e_1 = e_3$, $e_1 e_2 = e_3 + k e_4$, $e_2 e_2 = e_2 e_3 = -e_3 e_2 = e_4$.

(d′) $e_1 e_1 = e_1 e_2 = e_3$, $e_2 e_1 = -e_3 + a e_4$, $e_2 e_3 = e_4$, $e_3 e_1 = \lambda e_4$, $e_3 e_2 = \mu e_4$ $(a = 0, 1;\ \mu \neq 0)$.

(e′) $e_1 e_1 = -e_2 e_1 = e_3$, $e_1 e_2 = e_3 + e_4$, $e_2 e_3 = e_4$, $e_3 e_1 = \lambda e_4$, $e_3 e_2 = \mu e_4$.

(f′) $e_1 e_1 = -e_2 e_1 = e_3$, $e_1 e_2 = e_3 + a e_4$, $e_1 e_3 = k e_4$, $e_3 e_2 = e_4$ $(a = 0, 1;\ k \neq 0)$.

(g′) $e_1 e_1 = e_1 e_2 = e_3$, $e_2 e_1 = -e_3 + a e_4$, $e_3 e_1 = k e_4$, $e_2 e_3 = e_4$ $(a = 0, 1;\ k \neq 0)$.

(b) $\gamma_{123} - \gamma_{213} = 0$. ∞^1 classes.

(1) $e_1 e_1 = e_3$, $e_2 e_1 = a e_4$, $e_1 e_3 = e_4$, $e_3 e_1 = \lambda e_4$ $(a = 0, 1)$.

(2) $e_1 e_1 = e_3$, $e_1 e_2 = a e_4$, $e_3 e_1 = e_4$ $(a = 0, 1)$.

(3) $e_1 e_1 = e_3$, $e_3 e_1 = e_2 e_3 = -e_3 e_2 = e_4$.

(4) $e_1 e_1 = e_3$, $e_2 e_1 = a e_4$, $e_2 e_3 = -e_3 e_2 = e_4$ $(a = 0, 1)$.

(5) $e_1 e_1 = e_3$, $e_2 e_2 = e_2 e_3 = -e_3 e_2 = e_4$, $e_2 e_1 = a e_4$ $(a = 0, 1)$.

(6) $e_1 e_1 = e_3$, $e_2 e_2 = e_2 e_3 = -e_3 e_2 = e_3 e_1 = e_4$.

(7) $e_1 e_1 = e_3$, $e_2 e_1 = a e_4$, $e_2 e_2 = e_1 e_3 = e_4$, $e_3 e_1 = \lambda e_4$ $(a = 0, 1)$.

(8) $e_1 e_1 = e_3$, $e_2 e_2 = e_2 e_3 = -e_3 e_2 = e_4$, $e_2 e_1 = k e_4$.

(9) $e_1 e_1 = e_3$, $e_2 e_2 = e_2 e_3 = -e_3 e_2 = e_3 e_1 = e_4$.

(10) $e_1 e_1 = e_3$, $e_2 e_1 = a e_4$, $e_1 e_3 = b e_4$, $e_3 e_2 = e_4$ $(a, b = 0, 1)$.

(11) $e_1 e_1 = e_3$, $e_1 e_2 = a e_4$, $e_3 e_1 = b e_4$, $e_2 e_3 = e_4$, $e_3 e_2 = k e_4$ $(a, b = 0, 1)$.

III. $r = 2$.

(a) $\rho = 1$.

(1) $\prod_{i=1,2} (\gamma_{i34} + \gamma_{3i4}) \neq 0$. ∞^2 classes.

(a′) $e_2 e_1 = e_3$, $e_1 e_2 = k e_3 + a e_4$, $e_3 e_1 = e_4$, $e_3 e_2 = b e_4$, $e_i e_3 = \lambda e_3 e_i$ $(i = 1, 2)$, $(a, b = 0, 1;\ \lambda \neq -1;\ |k| \leq 1$ if $b = 1)$.

(b′) $e_2 e_1 = e_3$, $e_1 e_2 = k e_3 + a e_4$, $e_1 e_3 = e_4$, $e_2 e_3 = b e_4$ $(a, b = 0, 1;\ |k| \leq 1$ if $b = 1)$.

(2) $\prod_{i=1,2} (\gamma_{i34} + \gamma_{3i4}) = 0$. ∞^2 classes.

(a′) $e_1 e_1 = a e_4$, $e_1 e_2 = k e_4$, $e_2 e_1 = e_3$, $e_2 e_2 = b e_4$, $e_1 e_3 = -e_3 e_1 = e_4$ $(a, b = 0, 1)$.

(b′) $e_1 e_1 = a e_4$, $e_1 e_2 = k e_3$, $e_2 e_1 = e_3$, $e_2 e_2 = \lambda e_4$, $e_1 e_3 = -e_3 e_1 = e_4$, $e_2 e_3 = -e_3 e_2 = c e_4$

$(a, c = 0, 1;\ |k| \leq 1$ if $c = 1$, $a = \lambda = 0$ or if $a = c = 1$, $\lambda \neq 0)$.

(b) $\rho = 2$.

(1) $\prod_{i=1,2} (\gamma_{i34} + \gamma_{3i4}) \neq 0$. ∞^3 classes.

(a′) $e_2 e_1 = e_3$, $e_1 e_3 = e_3 e_2 = e_4$, $e_1 e_2 = k e_3 + a e_4$, $e_3 e_1 = \lambda e_4$ $(a = 0, 1)$.

(b′) $e_2 e_1 = e_3$, $e_1 e_3 = e_2 e_3 = e_4$, $e_1 e_2 = k e_3 + a e_4$, $e_3 e_1 = \lambda e_4$, $e_3 e_2 = \mu e_4$ $(|k| \leq 1;\ a = 0, 1)$.

(2) $\prod_{i=1,2} (\gamma_{i34} + \gamma_{3i4}) = 0$. ∞^3 classes.

(a′) $e_2 e_1 = e_3$, $e_1 e_2 = k e_3 + \lambda e_4$, $e_2 e_3 = -e_3 e_2 = e_1 e_3 = e_4$, $e_3 e_1 = \mu e_4$ $(|k| \leq 1)$.

(b′) $e_2 e_1 = e_3$, $e_1 e_2 = k e_3 + a e_4$, $e_2 e_3 = -e_3 e_2 = e_4$, $e_3 e_1 = e_4$ $(|k| \leq 1;\ a = 0, 1)$.

(c′) $e_2 e_1 = e_3$, $e_1 e_2 = k e_3 + \lambda e_4$, $e_2 e_2 = e_1 e_3 = e_2 e_3 = -e_3 e_2 = e_4$, $e_3 e_1 = \mu e_4$.

(d′) $e_2 e_1 = e_3$, $e_1 e_2 = k e_3 + a e_4$, $e_2 e_2 = e_3 e_1 = e_2 e_3 = -e_3 e_2 = e_4$ $(a = 0, 1)$.

Type E: $n' = n'' = 1$, $n^{(3)} = 2$; $\alpha = 4$.

$$e_1 e_1 = e_2,\quad e_1 e_2 = e_3,\quad e_2 e_1 = e_4.$$

Type F: $n' = n'' = n^{(3)} = n^{(4)} = 1$; $\alpha = 5$.

I. $\gamma_{224} = 0$. ∞^2 classes.

(a) $e_1 e_1 = e_2$, $e_1 e_2 = e_3$, $e_2 e_1 = k e_3 + a e_4$, $e_1 e_3 = e_4$, $e_3 e_1 = \mu e_4$ $(a = 0, 1)$.

(b) $e_1 e_1 = e_2$, $e_2 e_1 = e_3$, $e_1 e_3 = e_4$, $e_3 e_1 = \mu e_4$.

(c) $e_1 e_1 = e_2$, $e_1 e_2 = e_3$, $e_2 e_1 = k e_3 + a e_4$, $e_3 e_1 = e_4$ $(a = 0, 1)$.

(d) $e_1 e_1 = e_2$, $e_1 e_2 = a e_4$, $e_2 e_1 = e_3$, $e_3 e_1 = e_4$ $(a = 0, 1)$.

II. $\gamma_{224} \neq 0$. ∞^3 classes.

(a) $\gamma_{123} \gamma_{213} = 0$.

(1) $e_1 e_1 = e_2$, $e_2 e_1 = e_3$, $e_1 e_2 = a e_4$, $e_2 e_2 = e_4$, $e_1 e_3 = -e_4$, $e_3 e_1 = \mu e_4$.

(2) $e_1 e_1 = e_2$, $e_2 e_1 = e_3$, $e_2 e_2 = e_4$, $e_1 e_3 = \mu e_4$, $e_3 e_1 = \nu e_4$ $(\mu \neq -1)$.

(3) $e_1 e_1 = e_2$, $e_1 e_2 = e_3$, $e_2 e_1 = a e_4$, $e_2 e_2 = -e_3 e_1 = e_4$, $e_1 e_3 = \mu e_4$.

(4) $e_1 e_1 = e_2$, $e_1 e_2 = e_3$, $e_2 e_2 = e_4$, $e_3 e_1 = \mu e_4$, $e_1 e_3 = \nu e_4$ $(\mu \neq -1)$.

(b) $\gamma_{123} \gamma_{213} \neq 0$.

$e_1 e_1 = e_2$, $e_1 e_2 = e_3$, $e_2 e_1 = k e_3 + a e_4$, $e_2 e_2 = e_4$, $e_1 e_3 = \mu e_4$, $e_3 e_1 = \nu e_4$ $(k \neq 0;\ a = 0, 1)$.

The algebras of Type F are completely characterized by the value of $r_{0,0}(r_{1,0} - 1)$, where $r_{0,0}$ and $r_{1,0}$ are respectively the ranks of

$$\begin{pmatrix} c_1 & 0 \\ c_2 & 0 \end{pmatrix},\quad \begin{pmatrix} c_1 & 1 \\ c_2 & 0 \end{pmatrix}$$

$$\begin{cases} c_1 = \gamma_{123} \gamma_{214} - \gamma_{124} \gamma_{213}, \\ c_2 = (\gamma_{123} \gamma_{314} - \gamma_{213} \gamma_{134})(\gamma_{123} + \gamma_{213}) + (\gamma_{123} - \gamma_{213}) \gamma_{112} \gamma_{224}, \end{cases}$$

together with the ranks of the following matrices:

$$\begin{pmatrix}\gamma_{123} & 1\\ \gamma_{213} & \varkappa\end{pmatrix},\quad \begin{pmatrix}\gamma_{123} & 0\\ \gamma_{213} & 1\end{pmatrix},\quad \begin{pmatrix}\gamma_{134} & 1\\ \gamma_{314} & \mu\end{pmatrix},\quad \begin{pmatrix}\gamma_{134} & 0\\ \gamma_{314} & 1\end{pmatrix},$$

$$\begin{pmatrix}\gamma_{123}\,\gamma_{134} & \varkappa'\\ \gamma_{112}\,\gamma_{224} & 1\end{pmatrix},\quad \begin{pmatrix}\gamma_{123}\,\gamma_{134} & 1\\ \gamma_{112}\,\gamma_{224} & 0\end{pmatrix},\quad \begin{pmatrix}\gamma_{123}\,\gamma_{314} & \varkappa''\\ \gamma_{112}\,\gamma_{224} & 1\end{pmatrix},\quad \begin{pmatrix}\gamma_{123}\,\gamma_{314} & 1\\ \gamma_{112}\,\gamma_{224} & 0\end{pmatrix},$$

$$\begin{pmatrix}\gamma_{213}\,\gamma_{134} & \lambda'\\ \gamma_{112}\,\gamma_{224} & 1\end{pmatrix},\quad \begin{pmatrix}\gamma_{213}\,\gamma_{134} & 1\\ \gamma_{112}\,\gamma_{224} & 0\end{pmatrix},\quad \begin{pmatrix}\gamma_{213}\,\gamma_{314} & \lambda''\\ \gamma_{112}\,\gamma_{224} & 1\end{pmatrix},\quad \begin{pmatrix}\gamma_{213}\,\gamma_{314} & 1\\ \gamma_{112}\,\gamma_{224} & 0\end{pmatrix}.$$

Type G: $n'=2$, $n^{(2)}=1$, $n^{(3)}=0$, $n^{(4)}=1$.

I. $Q_3 \equiv \sum_{i,j}^{1,2} r_{ij3}x_ix_j$ of rank 0.

(1) $e_1e_2=-e_2e_1=e_3,\ e_3e_3=e_4.$

(2) $e_1e_2=-e_2e_1=e_3,\ e_3e_1=e_3e_3=e_4.$

(3) $e_1e_2=-e_2e_1=e_3,\ e_1e_1=e_3e_3=e_4,\ e_3e_1=\varkappa e_4.$

(4) $e_1e_2=-e_2e_1=e_3,\ e_1e_1=e_3e_2=e_3e_3=e_4.$

(5) $e_1e_2=e_3,\ e_2e_1=-e_3+e_4,\ e_3e_1=e_3e_3=e_4,\ e_3e_2=\varkappa e_4.$

(6) $e_1e_2=e_3,\ e_2e_1=-e_3+e_4,\ e_3e_3=e_4,\ e_3e_2=ae_4\,(a=0,\,1).$

II. Q_3 of rank 1.

(1) $e_1e_1=e_3,\ e_2e_1=e_2e_2=e_3e_3=e_4,\ e_3e_1=\varkappa e_4,\ e_3e_2=\lambda e_4.$

(2) $e_1e_1=e_3,\ e_2e_2=e_3e_3=e_4,\ e_3e_2=ae_4\,(a=0,\,1).$

(3) $e_1e_1=e_3,\ e_2e_2=e_3e_3=e_3e_1=e_4,\ e_3e_2=\varkappa e_4.$

(4) $e_1e_1=e_3,\ e_1e_2=e_3e_2=e_3e_3=e_4,\ e_2e_1=\varkappa e_4.$

(5) $e_1e_1=e_3,\ e_1e_2=e_3e_1=e_3e_3=e_4,\ e_2e_1=\varkappa e_4.$

(6) $e_1e_1=e_3,\ e_1e_2=e_3e_3=e_4,\ e_2e_1=\varkappa e_4.$

(7) $e_1e_1=e_3,\ e_2e_1=e_3e_2=e_3e_3=e_4.$

(8) $e_1e_1=e_3,\ e_2e_1=e_3e_1=e_3e_3=e_4.$

(9) $e_1e_1=e_3,\ e_2e_1=e_3e_3=e_4.$

(10) $e_1e_1=e_3,\ e_3e_2=e_3e_3=e_4.$

(11) $e_1e_1=e_3,\ e_3e_1=e_3e_3=e_4.$

(12) $e_1e_1=e_3,\ e_3e_3=e_4.$

(13) $e_1e_1=e_3,\ e_1e_2=e_3+\varkappa e_4,\ e_2e_1=-e_3+\lambda e_4,\ e_3e_2=e_3e_3=e_4.$

(14) $e_1e_1=e_1e_2=e_3,\ e_2e_1=-e_3+\varkappa e_4,\ e_2e_2=e_3e_3=e_4,\ e_3e_2=\lambda e_4.$

(15) $e_1e_1=e_1e_2=e_3,\ e_2e_1=-e_3+\varkappa e_4,\ e_3e_1=e_3e_3=e_4,\ e_2e_2=\lambda e_4.$

(16) $e_1e_1=e_3,\ e_1e_2=e_3+\varkappa e_4,\ e_2e_1=-e_3+\lambda e_4,\ e_3e_1=e_3e_3=e_4,\ e_2e_2=\mu e_4.$

(17) $e_1e_1=e_1e_2=e_3,\ e_2e_1=-e_3+\varkappa e_4,\ e_3e_3=e_2e_2=e_4.$

(18) $e_1e_1=e_1e_2=e_3,\ e_2e_1=-e_3+\varkappa e_4,\ e_3e_3=e_4.$

III. Q_3 of rank 2.

(1) $e_1e_2=e_3,\ e_2e_1=\varkappa e_3+\lambda e_4,\ e_1e_1=e_2e_2=e_3e_3=e_4,\ e_3e_1=\mu e_4,$
$e_3e_2=\nu e_4\,(\varkappa\neq-1).$

(2) $e_1e_2=e_3$, $e_2e_1=\varkappa e_3+\lambda e_4$, $e_1e_1=e_3e_2=e_3e_3=e_4$, $e_3e_1=\mu e_4$ $(\varkappa\neq-1)$.

(3) $e_1e_2=e_3$, $e_2e_1=\varkappa e_3+\lambda e_4$, $e_1e_1=e_3e_1=e_3e_3=e_4(\varkappa\neq-1)$.

(4) $e_1e_2=e_3$, $e_2e_1=\varkappa e_3+\lambda e_4$, $e_1e_1=e_3e_3=e_4(\varkappa\neq-1)$.

(5) $e_1e_2=e_3$, $e_2e_1=\varkappa e_3+\lambda e_4$, $e_3e_1=e_3e_2=e_3e_3=e_4(\varkappa\neq-1)$.

(6) $e_1e_2=e_3$, $e_2e_1=\varkappa e_3+ae_4$, $e_3e_1=e_3e_3=e_4(\varkappa\neq-1;\ a=0, 1)$.

(7) $e_1e_2=e_3$, $e_2e_1=\varkappa e_3+ae_4$, $e_3e_3=e_4(\varkappa\neq-1;\ a=0, 1)$.

Type H: $n'=n^{(2)}=n^{(3)}=1$, $n^{(4)}=0$, $n^{(5)}=1$.

(1) $e_1e_1=e_2$, $e_1e_2=e_3=-e_2e_1$, $e_2e_3=-e_3e_2=e_4$, $e_3e_1=ae_4(a=0, 1)$.

(2) $e_1e_1=e_2$, $e_1e_2=e_3$, $e_2e_1=-e_3+e_4$, $e_2e_3=-e_3e_2=e_4$, $e_3e_1=\varkappa e_4$.
Here there is a class for every value of $\varkappa^2$.

(3) $e_1e_1=e_2$, $e_1e_2=e_3$, $e_2e_1=\varkappa e_3$, $e_2e_3=-e_3e_2=e_4$, $e_3e_1=ae_4(a=0, 1)$.

(4) $e_1e_1=e_2$, $e_1e_2=e_3$, $e_2e_1=-e_3+\varkappa e_4$, $e_2e_2=e_2e_3=-e_3e_2=e_4$, $e_3e_1=\lambda e_4$.

(5) $e_1e_1=e_2$, $e_1e_2=e_3$, $e_2e_1=\varkappa e_3$, $e_2e_2=e_2e_3=-e_3e_2=e_4$, $e_3e_1=\lambda e_4$.

(6) $e_1e_1=e_2$, $e_1e_2=e_3$, $e_2e_1=\varkappa e_3$, $e_3e_2=e_3e_1=e_4$, $e_1e_3=ae_4(a=0, 1)$.

(7) $e_1e_1=e_2$, $e_1e_2=e_3$, $e_2e_1=\varkappa e_3$, $e_3e_2=e_4$, $e_1e_3=ae_4(a=0, 1)$.

(8) $e_1e_1=e_2$, $e_1e_2=e_3$, $e_2e_1=\varkappa e_3+e_4$, $e_3e_2=e_4$, $e_3e_1=\lambda e_4$, $e_1e_3=\mu e_4$.

(9) $e_1e_1=e_2$, $e_2e_1=e_3$, $e_3e_2=e_4$, $e_1e_3=ae_4(a=0, 1)$.

(10) $e_1e_1=e_2$, $e_2e_1=e_3$, $e_3e_1=e_3e_2=e_4$, $e_1e_3=\varkappa e_4$.

(11) $e_1e_1=e_2$, $e_1e_2=-e_2e_1=e_3$, $e_2e_3=e_4$, $e_3e_2=\varkappa e_4$, $e_3e_1=ae_4$
$(\varkappa\neq-1;\ a=0, 1)$.

(12) $e_1e_1=e_2$, $e_2e_1=e_3$, $e_1e_2=e_2e_3=e_4$, $e_1e_3=\varkappa e_4$, $e_3e_1=\lambda e_4$,
$e_3e_2=\mu e_4(\mu\neq-1)$.

(13) $e_1e_1=e_2$, $e_2e_1=e_3$, $e_2e_3=e_4$, $e_3e_2=\varkappa e_4$, $e_3e_1=ae_4(a=0, 1;\ \varkappa\neq-1)$.

(14) $e_1e_1=e_2$, $e_2e_1=e_3$, $e_1e_3=e_2e_3=e_4$, $e_3e_1=\varkappa e_4$, $e_3e_2=\lambda e_4(\lambda\neq-1)$.

(15) $e_1e_1=e_2$, $e_1e_2=e_3$, $e_2e_1=\varkappa e_3$, $e_2e_3=e_4$, $e_3e_2=\lambda e_4$, $e_3e_1=ae_4$
$(\lambda\neq-1;\ a=0, 1)$.

(16) $e_1e_1=e_2$, $e_1e_2=e_3$, $e_3e_1=\varkappa e_3$, $e_1e_3=e_2e_3=e_4$, $e_3e_1=\lambda e_4$, $e_3e_2=\mu e_4$
$(\mu\neq-1, 0;\ \varkappa\neq0)$.

(17) $e_1e_1=e_2$, $e_1e_2=e_3$, $e_2e_1=\varkappa e_3+\lambda e_4$, $e_2e_3=e_4$, $e_3e_2=\mu e_4$, $e_3e_1=ae_4$
$(a=0, 1;\ \lambda=1, \text{ if } a=0;\ \mu\neq-1, 0;\ \varkappa\neq0)$.

(18) $e_1e_1=e_2$, $e_1e_2=e_3$, $e_2e_1=\varkappa e_3+\lambda e_4$, $e_1e_3=e_2e_3=e_4$, $e_3e_1=\mu e_4$,
$e_3e_2=\nu e_4(\varkappa\neq0;\ \nu\neq-1, 0)$.

Type I: $n'=n^{(2)}=n^{(3)}=1$, $n^{(4)}=n^{(5)}=0$, $n^{(6)}=1$.

(1) $e_1e_1=e_2$, $e_1e_2=e_3$, $e_2e_1=\varkappa e_3+\lambda e_4$, $e_2e_2=e_3e_3=e_4$, $e_3e_1=\mu e_4$, $e_3e_2=\nu e_4$ $(\varkappa\neq-1)$. Here there is a class for every set of values of λ^2, μ^2, ν^2 and $\varkappa$.

(2) $e_1e_1=e_2$, $e_1e_2=e_3$, $e_2e_1=\varkappa e_3+e_4$, $e_3e_3=e_4$, $e_3e_1=\mu e_4$, $e_3e_2=\nu e_4(\varkappa\neq-1)$.
There is a class for every set of values of μ^3, ν^3 and $\varkappa$.

(3) $e_1e_1=e_2$, $e_1e_2=e_3$, $e_2e_1=\varkappa e_3$, $e_3e_1=e_3e_3=e_4$, $e_3e_2=\nu e_4 (\varkappa\neq -1)$. There is a class for every value of ν^2 and $\varkappa$.

(4) $e_1e_1=e_2$, $e_1e_2=e_3$, $e_3e_3=e_4$, $e_2e_1=\varkappa e_3$, $e_3e_2=ae_4 (a=0, 1;\ \varkappa\neq -1)$.

(5) $e_1e_1=e_2$, $e_1e_2=e_3$, $e_2e_1=-e_3+\varkappa e_4$, $e_2e_3=e_3e_3=e_4$, $e_2e_2=\lambda e_4$, $e_3e_1=\mu e_4$, $e_3e_2=\nu e_4$.

(6) $e_1e_1=e_2$, $e_1e_2=e_3$, $e_2e_1=-e_3+\varkappa e_4$, $e_3e_2=e_3e_3=e_4$, $e_2e_2=\lambda e_4$, $e_3e_1=\mu e_4$.

(7) $e_1e_1=e_2$, $e_1e_2=e_3$, $e_2e_1=-e_3+\varkappa e_4$, $e_2e_2=e_3e_3=e_4$, $e_3e_1=\lambda e_4$. There is a class here for every set of values of $\varkappa^2$, λ^2.

(8) $e_1e_1=e_2$, $e_1e_2=e_3$, $e_2e_1=-e_3+e_4$, $e_3e_3=e_4$, $e_3e_1=\varkappa e_4$. Here there is a class for every value of $\varkappa^3$.

(9) $e_1e_1=e_2$, $e_1e_2=-e_2e_1=e_3$, $e_3e_3=e_4$, $e_3e_1=ae_4 (a=0, 1)$.

(10) $e_1e_1=e_2$, $e_2e_1=e_3$, $e_2e_2=e_3e_3=e_4$, $e_1e_2=\lambda e_4$, $e_1e_3=\mu e_4$, $e_2e_3=\nu e_4$. There is a class for every set of values of λ^2, μ^2, ν^2.

(11) $e_1e_1=e_2$, $e_2e_1=e_3$, $e_1e_2=e_3e_3=e_4$, $e_1e_3=\mu e_4$, $e_2e_3=\nu e_4$. Here there is a class for every set of values of μ^3, ν^3.

(12) $e_1e_1=e_2$, $e_2e_1=e_3$, $e_1e_3=e_3e_3=e_4$, $e_2e_3=\nu e_4$. There is a class for every value of ν^2.

(13) $e_1e_1=e_2$, $e_2e_1=e_3$, $e_3e_3=e_4$, $e_2e_3=ae_4 (a=0, 1)$.

Type J: $n'=n^{(2)}=1$, $n^{(3)}=0$, $n^{(4)}=n^{(5)}=1$.

(1) $e_1e_1=e_2$, $e_2e_2=e_3$, $e_1e_3=e_4$, $e_2e_1=ae_4$, $e_3e_1=\varkappa e_4 (a=0, 1)$.

(2) $e_1e_1=e_2$, $e_2e_2=e_3$, $e_2e_1=e_3+\varkappa e_4$, $e_3e_1=\lambda e_4$.

(3) $e_1e_1=e_2$, $e_2e_2=e_3$, $e_2e_1=\varkappa e_3+\lambda e_4$, $e_1e_2=e_1e_3=e_4$, $e_3e_1=\mu e_4$. There is a class for every set of values of $\varkappa^2$, λ and μ.

(4) $e_1e_1=e_2$, $e_2e_2=e_3$, $e_3e_1=e_4$, $e_1e_2=ae_4 (a=0, 1)$.

(5) $e_1e_1=e_2$, $e_2e_2=e_3$, $e_1e_2=e_3+\varkappa e_4$, $e_3e_1=e_4$.

(6) $e_1e_1=e_2$, $e_2e_2=e_3$, $e_2e_1=e_3e_1=e_4$, $e_1e_2=\varkappa e_3+\lambda e_4$. There is a class here for every set of values of $\varkappa^2$, λ.

Type K: $n'=n^{(2)}=1$, $n^{(3)}=0$, $n^{(4)}=1$, $n^{(5)}=0$, $n^{(6)}=1$.

(1) $e_1e_1=e_2$, $e_2e_2=e_3$, $e_1e_2=ae_3$, $e_2e_1=\varkappa e_3+\lambda e_4$, $e_2e_3=e_4$, $e_3e_1=\mu e_4$, $e_3e_2=\nu e_4 (a=0, 1)$.

(2) $e_1e_1=e_2$, $e_2e_2=e_3$, $e_1e_2=ae_3+e_4$, $e_2e_1=\varkappa e_3+\lambda e_4$, $e_2e_3=e_4$, $e_3e_1=\mu e_4$. There is a class here for every set of values of a^3, $\varkappa^3$, μ^3, λ and ν.

(3) $e_1e_1=e_2$, $e_2e_2=e_3$, $e_1e_2=\varkappa e_3$, $e_2e_1=\lambda e_3+\mu e_4$, $e_3e_2=e_1e_3=e_4$.

(4) $e_1e_1=e_2$, $e_1e_2=e_2e_2=e_3$, $e_2e_1=\varkappa e_3+\lambda e_4$, $e_3e_2=e_4$.

(5) $e_1e_1=e_2$, $e_2e_2=e_3$, $e_2e_1=e_3+\varkappa e_4$, $e_3e_2=e_4$.

(6) $e_1e_1=e_2$, $e_2e_2=e_3$, $e_2e_1=ae_4$, $e_3e_2=e_4 (a=0, 1)$.

Type L: $n'=n^{(2)}=1$, $n^{(3)}=0$, $n^{(4)}=1$, $n^{(5)}=n^{(6)}=n^{(7)}=0$, $n^{(8)}=1$.

(1) $e_1e_1=e_2$, $e_2e_2=e_3$, $e_2e_1=\varkappa e_3+\lambda e_4$, $e_1e_2=e_4=e_3e_3$, $e_3e_1=\mu e_4$, $e_3e_2=\nu e_4$, $e_2e_3=\sigma e_4$. Here there is a class for every set of values of $\varkappa$, λ^5, μ^5, ν^5, σ^5.

(2) $e_1e_1=e_2$, $e_2e_2=e_3$, $e_2e_1=e_3+\varkappa e_4$, $e_3e_3=e_4$, $e_3e_1=\lambda e_4$, $e_3e_2=\mu e_4$, $e_2e_3=\nu e_4$.

(3) $e_1e_1=e_2$, $e_2e_2=e_3$, $e_2e_1=e_3e_3=e_4$, $e_3e_1=\varkappa e_4$, $e_3e_2=\lambda e_4$, $e_2e_3=\mu e_4$. There is a class here for every set of values of $\varkappa$, λ^3, μ^3.

(4) $e_1e_1=e_2$, $e_2e_2=e_3$, $e_3e_1=e_3e_3=e_4$, $e_2e_3=\varkappa e_4$, $e_3e_2=\lambda e_4$. There is a class for every pair of values of $\varkappa^3$, λ^3.

(5) $e_1e_1=e_2$, $e_2e_2=e_3$, $e_2e_3=e_3e_3=e_4$, $e_3e_2=\varkappa e_4$. There is a class for every value of $\varkappa^2$.

(6) $e_1e_1=e_2$, $e_2e_2=e_3$, $e_3e_3=e_4$, $e_3e_2=ae_4 (a=0, 1)$.

CHAPTER IV.

Application to Associative Nilpotent Algebras.

15. *Invariantive Characterization of Two Types of General Nilpotent Algebras of Index Three.* There are some types of nilpotent algebras in n units over the field C of ordinary complex numbers which are completely characterized by invariants in a rather interesting manner; and we will now turn our attention to several such types.

Consider a nilpotent algebra $E=(e_1, \ldots, e_n)$ of index three, where $E^2=(e_n)$. Then there are two interesting types of such algebras according as the fundamental quadratic

$$Q \equiv \sum_{i,j}^{1,n'} \gamma_{ijn} x_i x_j \eta_n \qquad (n'=n-1)$$

is of rank zero for general η_n, or of rank one. When Q is of rank zero, we know from §§ 10, 13, 14 that the rank of

$$|\gamma_{ijn}| \quad (i, j=1, \ldots, n'), \tag{15}$$

is an invariant for algebras of this type which completely characterizes them when $n'=2, 3$. By induction it can readily be shown that

Theorem. *Over the field of ordinary complex numbers the rank of the invariant* (15) *completely characterizes nilpotent algebras E of genus* $(3; n-1, 1)$ *in canonical form, when the fundamental quadratic is of rank zero.*

Similarly by induction from $n=3, 4$ we have the

Theorem. *Over the field of ordinary complex numbers the rank of the invariant* (15) *completely characterizes nilpotent algebras E of genus* $(3; n-1, 1)$ *in canonical form, where the fundamental quadratic is of rank one.*

16. *General Commutative Associative Nilpotent Algebras.* In view of the theorem of § 8, the problem of the equivalence of two commutative n-ary algebras E and E' over a field F of index three (therefore associative)—where n' is the order of $B\equiv E \pmod{E^2}$ and of $B'\equiv E' \pmod{E'^2}$—is the same as the

problem of equivalence of two families of quadratic forms (10), in the field F, in n' independent variables $x_1, \ldots, x_{n'}$ and with $n''=n-n'$ independent parameters $\eta_{n'+1}, \ldots, \eta_n$ entering homogeneously.

More generally, the theorem of § 8, together with the invariantive property of the invariants of the fundamental forms of the complex $B \equiv E \pmod{E^2}$, gives us the following

THEOREM. *If we consider over a field F the n-ary commutative, associative nilpotent algebras of genus $(\alpha;\, n', \ldots, n^{(\alpha-1)})$ in "special" canonical form*

$$E = \sum_{i=1}^{\alpha-1} B_i, \text{ where } B \equiv E \pmod{E^2},\ B_i \equiv B^i \pmod{E^{i+1}}, \tag{16}$$

with multiplication table

$$e_p e_q = \sum_{r=\nu^{(k+l-1)}+1}^{\nu^{(k+l)}} \gamma_{pqr} e_r \quad \text{for } e_p \text{ in } B_k,\ e_q \text{ in } B_l, \tag{17}$$

and where

$$\nu^{(i)} = \sum_{j=1}^{i} n^{(j)}, \quad \sum_{j=1}^{\alpha-1} n^{(j)} = n,$$

then these algebras are characterized completely by those invariants which completely characterize the following set of $\alpha-2$ families of forms over the field F in n' independent variables:

$$\left.\begin{aligned} F_i &\equiv \sum_{r=\nu^{(i-1)}+1}^{\nu^{(i)}} F_{ir}\eta_r \quad (i=2, \ldots, \alpha-1), \\ \text{where} \qquad F_{2r} &\equiv \sum_{j,\,k=1}^{n'} \gamma_{jkr} x_j x_k \quad (\nu^{(i-1)}+1 \leq r \leq \nu^{(i)}), \\ F_{3r} &\equiv \sum_{j,\,k,\,l}^{1,\,n'} \Big(\sum_{m=\nu'+1}^{\nu''} \gamma_{klm}\gamma_{jmr} \Big) x_j x_k x_l, \end{aligned}\right\} \tag{18}$$

and where, in general, F_p is the fundamental p-ic of the complex B. Here the η's are parameters independent of the x's.

17. *Commutative Associative Nilpotent Algebras in n Units with $\alpha=3$, $n''=1$.* In particular, by the theorem of § 16, the problem of the invariantive characterization of commutative nilpotent n-ary algebras E over the field C where $E^2=(e_n)$,

$$E:\ e_j e_i = e_i e_j = \gamma_{ijn} e_n \ (i, j=1, \ldots, n-1),\ e_n e_k = e_k e_n = 0 \ (k=1, \ldots, n), \tag{19}$$

is essentially the same as the problem of the invariantive characterization of the quadratic forms

$$Q \equiv \sum_{i,\,j}^{1,\,n-1} \gamma_{ijn} x_i x_j \tag{20}$$

in the field C, where the x's are independent variables—the coordinates of the

general number in E (mod E^2). Now if r is the rank of Q, there is a non-singular transformation on the x's in C which will carry Q into $\sum_{k=1}^{r} x_k^2$; and hence, since Q is a covariant for the algebras considered, and since the x's are contragredient to the units $e_1, \ldots, e_{n-1}$, and in view of the commutativity of E, there is a non-singular transformation on the units of E which will carry it into the algebra where $e_k e_k = e_n$ $(k \leq r)$ and where the products not written are zero. Thus we have the

THEOREM. *The n-ary nilpotent commutative algebras of index three where $E^2 = (e_n)$ are completely characterized by the rank of the fundamental quadratics* (20), *where the γ_{ijk} are defined by* (19).

18. *Commutative, Associative Nilpotent Algebras in n Units of Genus* $(3;\, n-2,\, 2)$. Furthermore, § 16 shows that in the field C the problem of the equivalence of two commutative nilpotent n-ary algebras E' and E'' of index three, where $E'^2 = (e'_{n-1}, e'_n)$ and $E''^2 = (e''_{n-1}, e''_n)$,

$$\left.\begin{aligned} E' : \; e'_i e'_j &= \sum_k^{1,2} \gamma'_{ijn'+k} e'_{n'+k} \quad (i, j = 1, \ldots, n' = n-2), \\ E'' : \; e''_i e''_j &= \sum_k^{1,2} \gamma''_{ijn'+k} e''_{n'+k} \quad (i, j = 1, \ldots, n' = n-2), \end{aligned}\right\} \qquad (21)$$

is essentially the same as the problem of the equivalence of the two families of quadratic forms

$$\left.\begin{aligned} Q' &\equiv \sum_k^{1,2} \sum_{i,j}^{1,n'} \gamma'_{ijn'+k} x'_i x'_j \eta'_{n'+k}, \\ Q'' &\equiv \sum_k^{1,2} \sum_{i,j}^{1,n'} \gamma''_{ijn'+k} x''_i x''_j \eta''_{n'+k} \end{aligned}\right\} \qquad (22)$$

in the field C. Here the x''s are independent variables—actually they are the coordinates of the general number in E' (mod E'^2)—and the η''s are parameters independent of the x''s. Similarly for the x'''s and η'''s.

By § 10 or § 11, a necessary condition that E' be equivalent to E'' is that the polynomials

$$\left.\begin{aligned} \Big|\sum_k^{1,2} \gamma'_{ijn'+k} \eta'_{n'+k}\Big| &= \prod_f (a'_f \eta'_{n-1} + b'_f \eta'_n)^{l'_f} \quad (i, j = 1, \ldots, n'), \\ \Big|\sum_k^{1,2} \gamma''_{ijn'+k} \eta''_{n'+k}\Big| &= \prod_f (a''_f \eta''_{n-1} + b''_f \eta''_n)^{l''_f} \quad (i, j = 1, \ldots, n') \end{aligned}\right\} \qquad (23)$$

be equivalent under a non-singular transformation on the parameters. Let

$$a'_f \eta'_{n-1} + b'_f \eta'_n \quad (f = 1, \ldots, t') \qquad (24')$$

and

$$a''_f \eta''_{n-1} + b''_f \eta''_n \quad (f = 1, \ldots, t'') \qquad (24'')$$

be the distinct linear factors of the two polynomials (23) respectively. With each linear factor there belong certain exponents $l'_{f,r}$ (or $l''_{f,r}$) which tell the power to which that expression is a factor common to all r-rowed minors ($r=1, \ldots, n'$).

If the two polynomials (23) are equivalent, then $t'=t''$ and a one-to-one correspondence can be set up between the distinct linear factors (24′) and (24″) in such a way that the multiplicities of the corresponding factors are equal. If, further, the algebra E' be equivalent to E'', then this correspondence can be set up in such a way that, for every r, if a particular factor (24′) occur exactly $l'_{f,r}$ times as a factor common to all r-rowed minors of the determinant (23′), then its corresponding factor (24″) will be a factor of all r-rowed minors of the determinant (23″) exactly $l'_{f,r}$ times.

Conversely, by the theory of elementary divisors, if such a correspondence can be set up between the distinct linear factors (24′) and the distinct linear factors (24″), the fundamental quadratic (22) for $B' \equiv E'$ (mod E'^2) is equivalent to the fundamental quadratic (22) for $B''=E''$ (mod E''^2) under a non-singular transformation on the x''s, combined with a non-singular transformation on the η''s. Thus, since the quadratics (22) are covariants for the algebras of genus (3; n—2, 2), and since the x''s are contragredient to the units $e'_1, \ldots, e'_{n-2}$, and in view of the commutativity, there is a non-singular transformation on the units which will carry E' into E''. Hence the following

THEOREM. *Over the field C, two n-ary commutative nilpotent algebras* (21) *of genus* (3; n—2, 2) *in canonical form are equivalent if and only if the determinants* (23) *are equivalent and such that the elementary divisors of one can be made to correspond to the elementary divisors of the other.*

10. *Commutative, Associative Nilpotent Algebras with $n<6$.* As in the foregoing sections of this chapter, we will consider algebras defined over the field of ordinary complex numbers. Products not written are zero.

$n=1$. There is only one class. $e_1^2=0$.

$n=2$. There are two classes characterized by their index.

$$e_1^2=e_2;\quad e_i e_j=0 \quad (i, j=1, 2).$$

$n=3$. (1) $e_i e_j=0$ $(i, j=1, 2, 3)$.

(2) $e_1 e_1=e_2 e_2=e_3$.

(3) $e_1 e_1=e_3$.

(4) $e_1 e_1=e_2$, $e_1 e_2=e_2 e_1=e_3$.

The characteristic invariants here are respectively

$$I_1 = \frac{(4-\alpha)(3-\alpha)}{2},$$
$$I_2 = (4-\alpha)(\alpha-2)(r-1),$$
$$I_3 = (4-\alpha)(\alpha-2)(2-r),$$
$$I_4 = \frac{(\alpha-3)(\alpha-2)}{2},$$

where α is the index and r is the rank of $Q_3 \equiv \sum_{i,j}^{1,2} \gamma_{ij3} x_i x_j$.

$n=4$. (1) $e_i e_j = 0$ $(i, j=1, 2, 3, 4)$.

(2) $e_1 e_1 = e_2 e_2 = e_3 e_3 = e_4$.

(3) $e_1 e_1 = e_2 e_2 = e_4$.

(4) $e_1 e_1 = e_4$.

(5) $e_1 e_1 = e_3$, $e_1 e_2 = e_2 e_1 = e_4$.

(6) $e_1 e_1 = e_2 e_2 = e_3$, $e_1 e_2 = e_2 e_1 = e_4$.

(7) $e_1 e_1 = e_3$, $e_1 e_3 = e_3 e_1 = e_4$.

(8) $e_1 e_1 = e_3$, $e_2 e_2 = e_1 e_3 = e_3 e_1 = e_4$.

(9) $e_1 e_1 = e_3$, $e_2 e_3 = e_3 e_2 = e_4$.

(10) $e_1 e_2 = e_2 e_1 = e_3$, $e_1 e_3 = e_3 e_1 = e_4$.

(11) $e_1 e_2 = e_2 e_1 = e_3$, $e_2 e_2 = e_1 e_3 = e_3 e_1 = e_4$.

(12) $e_1 e_2 = e_2 e_1 = e_3$, $e_1 e_3 = e_3 e_1 = e_2 e_3 = e_3 e_2 = e_4$.

(13) $e_1 e_1 = e_2$, $e_1 e_2 = e_2 e_1 = e_3$, $e_2 e_2 = e_1 e_3 = e_3 e_1 = e_4$.

The characteristic invariants for $n=4$ are respectively:

$$I_1 = \frac{(5-\alpha)(4-\alpha)(3-\alpha)}{3!},$$

$$I_2 = (n'-2)\frac{(5-\alpha)(4-\alpha)(\alpha-2)}{2}(\rho_3-2)(\rho_3-1),$$

$$I_3 = (n'-2)\frac{(5-\alpha)(4-\alpha)(\alpha-2)}{2}(3-\rho_3)(\rho_3-1),$$

$$I_4 = (n'-2)\frac{(5-\alpha)(4-\alpha)(\alpha-2)}{2}\,\frac{(3-\rho_3)(2-\rho_3)}{2},$$

$$I_5 = (3-n')\frac{(5-\alpha)(4-\alpha)(\alpha-2)}{2}(2-r_2),$$

$$I_6 = (3-n')\frac{(5-\alpha)(4-\alpha)(\alpha-2)}{2}(r_2-1),$$

$$I_7 = \frac{(5-\alpha)(\alpha-3)(\alpha-2)}{2}(3-r_3)(2-\rho_2),$$

$$I_8 = \frac{(5-\alpha)(\alpha-3)(\alpha-2)}{2}(r_3-2)(m-2)(2-\rho_2),$$

$$I_9 = \frac{(5-\alpha)(\alpha-3)(\alpha-2)}{2}(r_3-2)(3-m)(2-\rho_2),$$

$$I_{10} = \frac{(5-\alpha)(\alpha-3)(\alpha-2)}{2}(3-r_3)(\rho_2-1),$$

$$I_{11} = \frac{(5-\alpha)(\alpha-3)(\alpha-2)}{2}(r_3-2)(m-2)(\rho_2-1),$$

$$I_{12} = \frac{(5-\alpha)(\alpha-3)(\alpha-2)}{2}(r_3-2)(3-m)(\rho_2-1),$$

$$I_{13} = \frac{(\alpha-4)(\alpha-3)(\alpha-2)}{3!},$$

where α is the index, ρ_2 is the rank of $|\gamma_{ij3}|$ $(i, j=1, 2)$, ρ_3 is the rank of $|\gamma_{ij4}|$ $(i, j=1, 2, 3)$, r_2 is the rank of the discriminant of $|\sum_k^{3,4} \gamma_{ijk}\eta_k|$ $(i, j=1, 2)$, r_3 is the rank of the discriminant of $|\sum_k^{3,4} \gamma_{ijk}\eta_k|$ $(i, j=1, 2, 3)$, and m is the maximum multiplicity of a linear factor of $|\sum_k^{3,4} \gamma_{ijk}\eta_k|$ $(i, j=1, 2, 3)$.

$n=5$. (1) $e_i e_j=0$ $(i, j=1, \ldots, 5)$.

(2) $e_1 e_1=e_3$, $e_1 e_2=e_2 e_1=e_4$, $e_2 e_2=e_5$.

(3) $e_1 e_1=e_4$, $e_1 e_2=e_2 e_1=e_5$.

(4) $e_1 e_1=e_2 e_2=e_4$, $e_1 e_2=e_2 e_1=e_5$.

(5) $e_1 e_2=e_2 e_1=e_4$, $e_2 e_3=e_3 e_2=e_5$.

(6) $e_1 e_1=e_4$, $e_2 e_2=e_5$, $e_3 e_3=e_4+e_5$.

(7) $e_1 e_1=e_2 e_2=e_4$, $e_3 e_3=e_5$.

(8) $e_1 e_2=e_2 e_1=e_4$, $e_2 e_2=e_3 e_3=e_5$.

(9) $e_1 e_2=e_2 e_1=e_3 e_3=e_4$, $e_2 e_2=e_5$.

(10) $e_1 e_2=e_2 e_1=e_3 e_3=e_4$, $e_2 e_3=e_3 e_2=e_5$.

(11) $e_1 e_1=e_2 e_2=e_3 e_3=e_4 e_4=e_5$.

(12) $e_1 e_1=e_2 e_2=e_3 e_3=e_5$.

(13) $e_1 e_1=e_2 e_2=e_5$.

(14) $e_1 e_1=e_5$.

(15) $e_1 e_1=e_3$, $e_1 e_2=e_2 e_1=e_4$, $e_1 e_3=e_3 e_1=e_5$.

(16) $e_1 e_1=e_3$, $e_1 e_2=e_2 e_1=e_4$, $e_2 e_2=e_1 e_3=e_3 e_1=e_5$.

(17) $e_1 e_1=e_3$, $e_1 e_2=e_2 e_1=e_4$, $e_2 e_3=e_3 e_2=e_1 e_4=e_4 e_1=e_5$.

(18) $e_1 e_1=e_2 e_2=e_3$, $e_1 e_2=e_2 e_1=e_4$, $e_1 e_3=e_3 e_1=e_2 e_3=e_3 e_2=e_5$, $e_1 e_4=e_4 e_1=e_2 e_4=e_4 e_2=e_5$.

(19) $e_1 e_1=e_2 e_2=e_3$, $e_1 e_2=e_2 e_1=e_4$, $e_1 e_3=e_3 e_1=e_2 e_4=e_4 e_2=e_5$.

(20) $e_1 e_1=e_4$, $e_2 e_2=e_1 e_4=e_4 e_1=e_5$.

(21) $e_1 e_1 = e_4$, $e_2 e_2 = e_3 e_3 = e_1 e_4 = e_4 e_1 = e_5$.

(22) $e_1 e_1 = e_3$, $e_1 e_3 = e_3 e_1 = e_4$, $e_1 e_4 = e_4 e_1 = e_3 e_3 = e_5$.

(23) $e_1 e_1 = e_3$, $e_1 e_3 = e_3 e_1 = e_4$, $e_2 e_2 = e_1 e_4 = e_4 e_1 = e_3 e_3 = e_5$.

(24) $e_1 e_1 = e_3$, $e_2 e_2 = e_1 e_3 = e_3 e_1 = e_4$, $e_1 e_4 = e_4 e_1 = e_3 e_3 = e_5$.

(25) $e_1 e_1 = e_2$, $e_1 e_2 = e_2 e_1 = e_3$, $e_2 e_2 = e_1 e_3 = e_3 e_1 = e_4$,
$e_1 e_4 = e_4 e_1 = e_2 e_3 = e_3 e_2 = e_5$.

The characteristic invariants for these classes are respectively:

$$I_1 = \frac{(6-\alpha)(5-\alpha)(4-\alpha)(3-\alpha)}{4!},$$

$$I_2 = \frac{(6-\alpha)(5-\alpha)(4-\alpha)(\alpha-2)}{3!}\,\frac{(3-n')(4-n')}{2!},$$

$$I_3 = \frac{(6-\alpha)(5-\alpha)(4-\alpha)(\alpha-2)}{3!}\,(n'-2)(4-n')\tau_1(2-\tau_1)(3-r_3),$$

$$I_4 = \frac{(6-\alpha)(5-\alpha)(4-\alpha)(\alpha-2)}{3!}\,(n'-2)(4-n')\,\frac{(\tau_1-1)\tau_1}{2}\,(3-r_3),$$

$$I_5 = \frac{(6-\alpha)(5-\alpha)(4-\alpha)(\alpha-2)}{3!}\,(n'-2)(4-n')\,\frac{(2-\tau_1)(1-\tau_1)}{2}\,(3-r_3),$$

$$I_6 = \frac{(6-\alpha)(5-\alpha)(4-\alpha)(\alpha-2)}{3!}\,(n'-2)(4-n')(r_3-2)\,\frac{(3-m)(2-m)}{2},$$

$$I_7 = \frac{(6-\alpha)(5-\alpha)(4-\alpha)(\alpha-2)}{3!}\,(n'-2)(4-n')(r_3-2)(3-m)(m-1)(2-\tau_2),$$

$$I_8 = \frac{(6-\alpha)(5-\alpha)(4-\alpha)(\alpha-2)}{3!}\,(n'-2)(4-n')(r_3-2)(3-m)(m-1)(\tau_2-1),$$

$$I_9 = \frac{(6-\alpha)(5-\alpha)(4-\alpha)(\alpha-2)}{3!}\,(n'-2)(4-n')(r_3-2)\,\frac{(m-2)(m-1)}{2}\,(\tau_2-1),$$

$$I_{10} = \frac{(6-\alpha)(5-\alpha)(4-\alpha)(\alpha-2)}{3!}\,(n'-2)(4-n')(r_3-2)\,\frac{(m-2)(m-1)}{2}\,\tau_2,$$

$$I_{11} = \frac{(6-\alpha)(5-\alpha)(4-\alpha)(\alpha-2)}{3!}\,\frac{(n'-3)(n'-2)}{2}\,\frac{(\rho-3)(\rho-2)(\rho-1)}{3!},$$

$$I_{12} = \frac{(6-\alpha)(5-\alpha)(4-\alpha)(\alpha-2)}{3!}\,\frac{(n'-3)(n'-2)}{2}\,\frac{(4-\rho)(\rho-2)(\rho-1)}{2},$$

$$I_{13} = \frac{(6-\alpha)(5-\alpha)(4-\alpha)(\alpha-2)}{3!}\,\frac{(n'-3)(n'-2)}{2}\,\frac{(4-\rho)(3-\rho)(\rho-1)}{2},$$

$$I_{14} = \frac{(6-\alpha)(5-\alpha)(4-\alpha)(\alpha-2)}{3!}\,\frac{(n'-3)(n'-2)}{2}\,\frac{(4-\rho)(3-\rho)(2-\rho)}{3!},$$

$$I_{15} = \frac{(6-\alpha)(5-\alpha)(\alpha-3)(\alpha-2)}{2^2}\,(3-n')(2-\rho')\,\frac{(3-r_4)(4-r_4)}{2},$$

$$I_{16}=\frac{(6-\alpha)(5-\alpha)(\alpha-3)(\alpha-2)}{2^2}(3-n')(2-\rho')(r_4-2)(4-r_4),$$

$$I_{17}=\frac{(6-\alpha)(5-\alpha)(\alpha-3)(\alpha-2)}{2^2}(3-n')(2-\rho')\frac{(r_4-3)(r_4-2)}{2},$$

$$I_{18}=\frac{(6-\alpha)(5-\alpha)(\alpha-3)(\alpha-2)}{2^2}(3-n')(\rho'-1)(4-r_4),$$

$$I_{19}=\frac{(6-\alpha)(5-\alpha)(\alpha-3)(\alpha-2)}{2^2}(3-n')(\rho'-1)(r_4-3),$$

$$I_{20}=\frac{(6-\alpha)(5-\alpha)(\alpha-3)(\alpha-2)}{2^2}(n'-2)(4-r_4),$$

$$I_{21}=\frac{(6-\alpha)(5-\alpha)(\alpha-3)(\alpha-2)}{2^2}(n'-2)(r_4-3),$$

$$I_{22}=\frac{(6-\alpha)(\alpha-4)(\alpha-3)(\alpha-2)}{3\,!}(3-r_3)(4-r_4),$$

$$I_{23}=\frac{(6-\alpha)(\alpha-4)(\alpha-3)(\alpha-2)}{3\,!}(3-r_3)(r_4-3),$$

$$I_{24}=\frac{(6-\alpha)(\alpha-4)(\alpha-3)(\alpha-2)}{3\,!}(r_3-2),$$

$$I_{25}=\frac{(\alpha-5)(\alpha-4)(\alpha-3)(\alpha-2)}{4\,!},$$

where r_3 is the rank of $|\sum_k^{4,5}\gamma_{ijk}\eta_k|$ $(i, j=1, 2, 3)$ and τ_i is the number of linearly independent sets of values (η_4, η_5) such that this determinant is of rank i; ρ_1 is the rank of $|\gamma_{ij5}|$ $(i, j=1, \ldots, 4)$; r_4 is the rank of $|\sum_k^{3,4,5}\gamma_{ijk}\eta_k|$ $(i, j=1,2,3,4)$; r_3 is the rank of $|\sum_k^{3,4}\gamma_{ijk}\eta_k|$ $(i, j=1, 2, 3)$; ρ' is the rank of the discriminant of $|\sum_k^{3,4}\gamma_{ijk}\eta_k|$ $(i, j=1, 2)$; and m is the maximum multiplicity of the linear factors of $|\sum_k^{4,5}\gamma_{ijk}\eta_k|$ $(i, j=1, 2, 3)$.

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